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A brief survey on residue theory of holomorphic foliations

Fernando Lourenço[1] and **Fernando Reis** [2]

[1]UFLA, Campus Universitário, Lavras-MG, Brazil
[2]UFES, Centro Universitário Norte do Espírito Santo, São Mateus-ES, Brazil

Abstract. This is a survey paper dealing with holomorphic foliations, with emphasis on Residue Theory and its applications. We start recalling the definition of holomorphic foliations as a subsheaf of the tangent sheaf of a manifold. The theory of Characteristic Classes of vector bundles is approached from this perspective. We define Chern Class of holomorphic foliations using the Chern-Weil theory and we remark that the Baum-Bott residue is a great tool that help us to classify some foliations. We present along the survey several recent results and advances in residue theory. We finish the work present some applications of residues to solve for example the Poincaré problem and the existence of minimal sets for foliations.

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1 Introduction

The residue theory of holomorphic foliations started with the work of P.Baum and R.Bott [7] in 1972. In their article the authors have devel-

e-mail: fernando.lourenco@ufla.br
e-mail: fernando.reis@ufes

oped a class for foliations associated with its singular set using differential geometry based on the Bott vanishing theorem. However this class, called residue, is only an element in the homology group. The question that arises is "how to calculate the residue?". We consider a holomorphic foliation $\mathcal{F}$ of dimension q on a complex manifold M of dimension n. If we consider φ a homogeneous symmetric polynomial of degree d, then to each compact connect component $Z \in \mathrm{Sing}(\mathcal{F})$ of the singular set of the foliation $\mathcal{F}$, there exists the homology class $\mathrm{Res}_\varphi(\mathcal{F}; Z)$ into the group $H_{2(n-d)}(Z;\mathbb{C})$ such that over certain condition on M, one has

$$\varphi(\mathcal{N}_{\mathcal{F}}) \frown [M] = \sum \mathrm{Res}_\varphi(\mathcal{F}; Z),$$

where $\mathcal{N}_{\mathcal{F}}$ represents the normal sheaf of the foliation $\mathcal{F}$.

This survey address the problem of computing this residue $\mathrm{Res}_\varphi(\mathcal{F}; Z)$ in some cases. In one complex variable, we have the Cauchy's residue of a holomorphic function and the Cauchy integral formula which help us to calculate it. On the other hand, in several complex variable and as a generalization of the Cauchy's residue we have the Grothendieck residue associated with a meromorphic form. Let $f = (f_1, \ldots, f_n) : U \subset \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a finite holomorphic map, such that $f(0) = 0$, and g be a holomorphic function on U. We define the Grothendieck residue by

$$\mathrm{Res}_0(g, f) = \Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n \int_\gamma \frac{g(z)dz_1 \wedge \ldots \wedge dz_n}{f_1 \cdots f_n}$$

where γ is a n-cycle with orientation prescribed by the n-form $d(\arg(f_1)) \wedge \cdots \wedge d(\arg(f_n)) \geq 0$. If we denote the merophorfic n-form $\dfrac{g(z)dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n}$ by ω we may use the notation

$$\mathrm{Res}_0(g, f) = \mathrm{Res}_0 \begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix}. \tag{1.1}$$

We observe that for $n = 1$, this residue is just the Cauchy's residue. Baum and Bott [7] also shown how to calculate the residue of a higher dimension foliation since certain assumptions are required in each irreducible component of the singular set of foliation. Take Z an irreducible component of $\mathrm{Sing}(\mathcal{F})$ with $\dim Z = \dim(\mathcal{F}) - 1$ and other generic hypotheses, one has

$$\mathrm{Res}_\varphi(\mathcal{F}; Z) = \mathrm{Res}_\varphi(\mathcal{F}|_{B_p}; p)[Z],$$

where $\mathrm{Res}_\varphi(\mathcal{F}|_{B_p}; p)$ is a certain Grothendieck residue.

Still in the same work [7] Baum-Bott show that the residue of a dimension one foliation at an isolated singular point can be also expressed by the Grothendieck residue in (1.1), where $f_1, \ldots, f_n$ are the components of the vector fields that induces locally $\mathcal{F}$.

In 1984, T. Suwa in [48] considers a foliation of complete intersection type and express a certain class of residues in terms of the Chern classes and the local Chern classes of the sheaf $\mathcal{E}xt^1_{\mathcal{O}_M}(\Omega_{\mathcal{F}}, \mathcal{O}_M)$. As an application, in 3.8 Corollary he gives a partial answer to Rationality Conjecture (see [7], p. 287). As another consequence in the case that the foliation has codimension one, he shows how to calculate residues at isolated singularities.

T. Suwa in [50] has developed a residue theory to distributions, where the localization considered there arises from a rather primitive fact, i.e., the Chern forms of degree greater than the rank of the vector bundle vanish. Hence the involutivity has nothing to do with it. For this reason the localization can be defined by rank reason, of some characteristic classes and associated residues of the normal sheaf of the distribution. Also in [50] (Proposition 4.4 p.15), the author shows, in particular, when the distribution is involutive (i.e., a foliation) the residue to distribution coincides with the corresponding Baum-Bott residues of foliation.

Years later, in 2005 T. Suwa and F. Bracci in [9], based in the classical Camacho-Sad residue (or index) theorem, have developed a residue theory for adequate singular pairs, which generalizes in certain way the classical Camacho-Sad residue. The authors show a Bott vanishing theorem and adapt the theory of Cech-de Rham theory and localization for adequate singular pair and prove that there exists the residue in this situation.

More recently, in 2015, F. Bracci and T. Suwa in [10] provide another effective way to compute residues. The authors consider a deformation of a complex manifold M, denoted by $\tilde{M} = \{M_t\}_{t\in\hat{P}}$, where the parameter space $\hat{P}$ is a $\mathcal{C}^\infty$ manifold and a deformation of a holomorphic foliation $\mathcal{F}$ of M, denoted by $\tilde{\mathcal{F}} = \{\mathcal{F}_t\}$. For all parameter $t \in \hat{P}$ they assume that the singular set S_t is compact. Let φ be a homogeneous symmetric polynomial of degree d and $\mathrm{Res}_\varphi(\mathcal{F}_t; S_t^\lambda)$ the Baum-Bott residue of $\mathcal{F}_t$ at the connect compact set S_t^λ. It is proved that the Baum-Bott residue continuously varies under this smooth deformation, i.e,

$$\lim_{t\to t_0} \sum_\lambda \mathrm{Res}_\varphi(\mathcal{F}_t; S_t^\lambda) = \mathrm{Res}_\varphi(\mathcal{F}_{t_0}; S_{t_0}^\lambda).$$

Subsequentely, the authors consider $\mathcal{F}$ as a germ at $0 \in \mathbb{C}^n$ of a simple almost Liouvillian foliation of codimension one and V a divisor of poles. Then it is shown that the residue of $\mathcal{F}$ at Z, which is an irreducible component of singular set of $\mathcal{F}$ of codimension 2, can be written as a sum (over the irreducible components of V that contains Z) in terms of Lehmann-Suwa residues [39]. This represents an effective way to compute residues:

$$BB(\mathcal{F}; Z) = \sum_{j=1}^{k} \mathrm{Res}(\gamma_0, V_j) Var(\mathcal{F}, V_j; Z).$$

In the paper [53] the author proves a more slight generalization of the Bott residue theorem to holomorphic foliations of dimension one. The proof is based on a localization formula of Duistermaat and Heckman type, wich has been discussed first in [8].

Recently in [29] in 2016 Corrêa et al. have studied several residue formulas for vector fields on compact complex orbifolds with isolated singularities, that is a special type of a singular variety.

It is worth remark that there are other types of residues and invariants associated to a foliation, as residues of logarithmic vector fields in [24], Camacho-Sad index in [16] and GSV-index of foliations and Pfaff systems in [52, 25].

We have finished the survey talking about the recent theme: residue to flags of holomorphic foliations and distributions. In the work [37], it was developed a general theory of residue to flags consisted of two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, where the first one is tangent the second one. There are many topics which are closely related to this flags and naturally appear in the theory of foliation. For example, a conjecture due to Marco Brunella (see [18], p.443) says that a two-dimensional holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^3$ either admits an invariant algebraic surface or it compose a flag of holomorphic foliations.

2 Chern-Weil Theory of Characteristic classes of holomorphic foliations

2.1 Holomorphic foliations

Let us begin by recalling the basic definition of singular holomorphic foliations and distributions. Let M be a complex manifold of dimension n and denote by Θ_M and Ω_M respectively, the sheaves of germs of holomorphic vector fields and holomorphic 1-forms on M. There are two definitions of singular foliations that turn out to be equivalent as long as we consider only reduced foliations. For this section about foliations theory we refer to [7, 37, 41, 49, 50].

A singular holomorphic distribution of dimension q on M is a coherent subsheaf $\mathcal{F}$ of Θ_M of rank q. Moreover, if $\mathcal{F}$ satisfies the following integrability condition

$$[\mathcal{F}_x, \mathcal{F}_x] \subset \mathcal{F}_x \quad \text{for all} \quad x \in M,$$

we say that $\mathcal{F}$ is a holomorphic foliation. The normal sheaf of $\mathcal{F}$ is defined as the quotient sheaf $\mathcal{N}_{\mathcal{F}} := \Theta_M/\mathcal{F}$, such that it is torsion free (it means that $\mathcal{F}$ is saturated). With this definition we have the following exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \Theta_M \longrightarrow \mathcal{N}_{\mathcal{F}} \longrightarrow 0.$$

We define the singular set of the distribution $\mathcal{F}$ by

$$\mathrm{Sing}(\mathcal{F}) := \mathrm{Sing}(\mathcal{N}_{\mathcal{F}}) = \{p \in M; \mathcal{N}_{\mathcal{F},p} \text{ is not locally free}\}.$$

We assume that $\mathrm{codim}(\mathrm{Sing}(\mathcal{F})) \geq 2$.

For the second one definition, a singular distribution $\mathcal{G}$ can be defined, as a dual way by means of differential forms, i. e., as a coherent subsheaf of Ω_M. If $\mathcal{G}$ satisfies the integrability condition, i.e.,

$$d\mathcal{G}_x \subset (\Omega_M \wedge \mathcal{G})_x \quad \text{for all } x \in M \setminus \mathrm{Sing}(\mathcal{G}),$$

we say that $\mathcal{G}$ is a foliation, where $\mathrm{Sing}(\mathcal{G}) := \mathrm{Sing}(\Omega_M/\mathcal{G})$.

The two definitions of foliations are equivalents and related by taking the annihilator of each other. If $\mathcal{F}$ is a foliation on M of dimension q, its annihilator is defined by

$$\mathcal{F}^a = \{v \in \Omega_M; < v, \omega >= 0 \quad \text{for all } \omega \in \mathcal{F}\}.$$

We say that $\mathcal{F}$ is reduced, if for any open set U in M,

$$\Gamma(U, \Theta_M) \cap \Gamma(U \setminus \mathrm{Sing}(\mathcal{F}), \mathcal{F}) = \Gamma(U, \mathcal{F}).$$

If we consider only reduced foliation, then $\mathcal{G} = \mathcal{F}^a$ and the converse is also true (see [49]).

To finished this subsection we present the follow definition.

Definition 2.1. Let V be an analytic subspace of a complex manifold X. We say that V is invariant by a foliation $\mathcal{F}$ if

$$T\mathcal{F}|_V \subset (\Omega^1_V)^*.$$

In particular cases,

- if V is a hypersurface we say that $\mathcal{F}$ is *logarithmic along* $\mathcal{F}$;
- if V is a reduced complete intersection of dimension $n - k$, defined by intersection of k hypersurfaces we say that $\mathcal{F}$ is *multlogarithmic along* $\mathcal{F}$.

2.1.1 Flag of holomorphic foliations

In this subsection we should define flag of foliations and show its main properties. For more details we refer to [37, 23, 41].

A flag of singular holomorphic foliations on a complex manifold M of dimension n, can be define by a finite sequence of k foliations $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_k)$ such that, outside of the singular sets, each foliation $\mathcal{F}_{i+1}$ is a subfoliation of $\mathcal{F}_i$ and we denote $\mathcal{F}_i \subset \mathcal{F}_{i+1}$, for each $i = 1, \dots, k-1$. In a more formal manner, for $k = 2$ one has the following.

Definition 2.2. Let $\mathcal{F}_1, \mathcal{F}_2$ be two holomorphic foliations on M of dimensions $q = (q_1, q_2)$. We say that $\mathcal{F} := (\mathcal{F}_1, \mathcal{F}_2)$ is a 2-flag of holomorphic foliations if $\mathcal{F}_1$ is a coherent sub $\mathcal{O}_M$-module of $\mathcal{F}_2$.

We note that, for $x \in M \setminus \cup_{i=1}^{2} \mathrm{Sing}(\mathcal{F}_i)$ the inclusion relation $T_x\mathcal{F}_1 \subset T_x\mathcal{F}_2$ holds, namely that the leaves of $\mathcal{F}_1$ are contained in leaves of $\mathcal{F}_2$. Here $T\mathcal{F}_i$ represents the tangent sheaf of the foliation $\mathcal{F}_i$, but throughout the text we will abuse of notation and denote it simply by $\mathcal{F}_i$. Now we observe that we have a diagram of short exact sequences of sheaves, called "turtle diagram".

$$\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[dl] & & 0 \\
& \mathcal{F}_1 \arrow[dr] \arrow[dd] & & \mathcal{N}_2 \arrow[ul] \arrow[ur] & \\
& & \Theta_M \arrow[ur] \arrow[dr] & & \\
& \mathcal{F}_2 \arrow[ur] \arrow[dr] & & \mathcal{N}_1 \arrow[uu] \arrow[dr] & \\
0 \arrow[ur] & & \mathcal{N}_{12} \arrow[ur] \arrow[dr] & & 0 \\
& 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

We define and use the following notation, let $\mathrm{Sing}(\mathcal{F})$ be the singular set of the flag $\mathcal{F}$ builds by the analytic set $\mathrm{Sing}(\mathcal{F}_1) \cup \mathrm{Sing}(\mathcal{F}_2)$ and $\mathcal{N}_{\mathcal{F}} := \mathcal{N}_{12} \oplus \mathcal{N}_2$ the normal sheaf of the flag, where $\mathcal{N}_{12}$ is the relative quotient sheaf $\mathcal{F}_2/\mathcal{F}_1$.

2.2 Characteristic classes via Chern-Weil theory

In this section we review the basic tools of Chern-Weil theory for working with residue and characteristic classes to vector bundles and sheaves. The residue theory of foliation was first introduced by Baum and Bott using differential geometry, in a series of papers ([7, 6, 5]). Later Lehman and Suwa in the decades of 1980 and 1990 present a new approach of residue theory using Chern-Weil theory (see [39]).

Definition 2.3. A connection for a complex vector bundle E of rank r on M is a $\mathbb{C}$-linear map

$$\nabla : A^0(M,E) \longrightarrow A^1(M,E)$$

that satisfies

$$\nabla(f.s) = df \otimes s + f.\nabla(s) \quad \text{for} \quad f \in A^0(M) \quad \text{and} \quad s \in A^0(M,E).$$

If ∇ is a connection for E, then it induces a $\mathbb{C}$-linear map

$$\nabla := \nabla^2 : A^1(M,E) \longrightarrow A^2(M,E)$$

satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s), \quad \omega \in A^1(M), \quad s \in A^0(M,E).$$

We define the composition map $K := \nabla \circ \nabla$ from $A^0(M,E)$ to $A^2(M,E)$ as the curvature of the connection ∇. If $s = (s_1, \ldots, s_n)$ is a frame of E on an open set U we have $\theta = (\theta_{ij})$ the connection matrix (where θ_{ij} are 1-forms on U) of E with respect to frame s. In the same way, we can get $K = (k_{ij})$ the curvature matrix of E with respect to s. If we consider $\sigma_i, i = 1, \ldots, r$ the i-th elementary symmetric functions in the eigenvalues of the matrix K

$$\det(It + K) = 1 + \sigma_1(K)t + \sigma_2(K)t^2 + \cdots + \sigma_r(K)t^r,$$

we may define a $2i$-form of Chern c_i on U by

$$c_i(K) := \sigma_i(\frac{\sqrt{-1}}{2\pi}K).$$

In general, if φ is a homogeneous symmetric polynomial in r variables of degree d, we may write $\varphi = \tilde{P}(c_1, \ldots, c_r)$ for some polynomial $\tilde{P}$. Then we can define

$$\varphi(K) := \tilde{P}(c_1(K), \ldots, c_r(K))$$

which is a closed form on M. Therefore, we have a cohomology class of E on M, $\varphi(E) := \varphi(K) \in H^{2d}(M; \mathbb{C})$.

Let $\mathcal{G}$ be a sheaf on M, S a compact connected set of M and U a relatively compact open neighborhood of S in M. We may consider $\mathcal{U} = \{U_0, U_1\}$ a covering of U, where $U_1 = U$ and $U_0 = U \setminus S$ and since there exists [3] the following resolution of $\mathcal{G}$

$$0 \longrightarrow \mathcal{A}_U(E_r) \longrightarrow \cdots \longrightarrow \mathcal{A}_U(E_0) \longrightarrow \mathcal{A}_U \otimes_{\mathcal{O}_U} \mathcal{G} \longrightarrow 0,$$

we can define the characteristic class $\varphi(\mathcal{G})$ on U using the *virtual bundle* $\xi := \sum_{i=0}^{r}(-1)^i E_i$, i. e., $\varphi(\mathcal{G}) := \varphi(\xi)$.

Given $\mathcal{F}$ a holomorphic foliation on M and φ a homogeneous symmetric polynomial of degree d, one has the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \Theta_M \longrightarrow \mathcal{N}_{\mathcal{F}} \longrightarrow 0.$$

Then $\varphi(\mathcal{N}_{\mathcal{F}})$ denotes the characteristic class of $\mathcal{F}$ and it is an element in cohomology group $H^{2d}(M; \mathbb{C})$. We denote by P the Poincaré homomorphism (or isomorphism if M is nonsingular) from $H^{2d}(M; \mathbb{C})$ to $H_{2(n-d)}(M; \mathbb{C})$ and by A the Alexander homomorphism (or isomorphism if S is nonsingular) $A : H^{2d}(M, M\backslash S; \mathbb{C}) \longrightarrow H_{2(n-d)}(S; \mathbb{C})$ we have the following commuting diagram:

$$\begin{tikzcd} H^{2d}(M, M\backslash S; \mathbb{C}) \arrow[r] \arrow[d, "A"'] & H^{2d}(M; \mathbb{C}) \\ H_{2(n-d)}(S; \mathbb{C}) \arrow[r, "i^*"] & H_{2(n-d)}(M; \mathbb{C}) \arrow[u, "P"'] \end{tikzcd}$$

where this map $H^{2d}(M, M\backslash S; \mathbb{C}) \longrightarrow H^{2d}(M; \mathbb{C})$ represents a lift that can be interpreted, in terms of foliation theory, by the Bott vanishing theorem (see [49], Theorem 9.11). Thus we have the residue of foliation $\mathcal{F}$, denoted by $\mathrm{Res}_{\varphi}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}; S)$ in $H_{2(n-d)}(S; \mathbb{C})$ as the image of $\varphi_S(\mathcal{N}_{\mathcal{F}}; \mathbb{C}) \in H^{2d}(M, M\backslash S; \mathbb{C})$ by the Alexander duality which is independent of all choices involved.

In general it is not possible to calculate such residue directly, namely, by the above definition. It is then important when one can calculate such element using tools like differential geometry, foliation theory, complex analysis or singularities theory. The goal of this survey is to present some results in this direction.

2.3 Some results about residues of holomorphic foliations

The residue theory of holomorphic foliations was developed by several authors in the last years, we cite for instance, Baum and Bott in [7, 5], Brasselet and Suwa in [11] and Bracci and Suwa in [9] and [10].

We would like to make a special reference to book [36], it is dedicated to deep study of indices associated to vector fields in isolated singularity, in the cases where the underlying space is either a smooth variety or a singular variety. The authors defines several notions of index such that: PoincarÃ©-Hopt index, Schwartz index, GSV index, Virtual index, Homological index and others. This is a very valuable reference for those interested in the subject.

This subject plays a major role in several areas of mathematics and is of great interest to other sciences.

This book is dedicated to the study of indices of vector fields and flows around an isolated singularity, or stationary point, in the cases where the underlying space is either a manifold or a singular variety. This subject plays a major role in several areas of mathematics and is of great interest to other sciences.

In this section we take into account recent results obtained by various authors that emerged in the recent years ago related to residues of holomorphic foliations.

We begin this section with the classical Grothendieck residue, that for more details we refer to [35, 49, 47]. Let us take a germ ω at 0 of holomorphic n-form, a neighborhood U of 0 in $\mathbb{C}^n$ and $a_1, \ldots, a_n$ germs of holomorphic functions such that $V(a_1, \ldots, a_n) = \{0\}$. The Grothendieck residue of ω at 0 is defined by

$$\mathrm{Res}_0 \left[\begin{array}{c} \omega \\ a_1, \ldots, a_n \end{array} \right] = \frac{1}{(2\pi\sqrt{-1})^n} \int_\gamma \frac{\omega}{a_1 \cdots a_n},$$

where γ is a n-cycle in U defined by

$$\gamma = \{z \in U; |a_1(z)| = \cdots = |a_n(z)| = \epsilon\}$$

and oriented by $d(\arg(a_1)) \wedge \cdots \wedge d(\arg(a_n)) \geq 0$. We remark that the above residue is the usual Cauchy residue at 0 of the meromorphic 1-form ω/a_1 when $n = 1$.

In order to formulate the first important result in theory foliation, that relates the Baum-Bott residue of certain foliations to the Grothendieck residue, let $\mathcal{F}$ be a holomorphic foliation of dimension one in a complex compact manifold M of dimension n. We assume that $\mathcal{F}$ has only isolated singularities. Let φ be a homogeneous symmetric polynomial of degree n and $p \in \mathrm{Sing}(\mathcal{F})$ an isolated point. Then

Theorem 2.4. *([7], Theorem 1)*

$$\mathrm{Res}_\varphi(\mathcal{F}; p) = \mathrm{Res}_p \left[\begin{array}{c} \varphi(JX) \\ X_1, \ldots, X_n \end{array} \right],$$

where $X = (X_1, \ldots, X_n)$ is a germ of a holomorphic vector field at p, local representative of $\mathcal{F}$.

In [48], (3.12) Proposition, T. Suwa considers $\mathcal{F}$ a holomorphic distribution (not necessarily involutive) of codimension one and taking 0 (it can be another point p) as an isolated singularity of this distribution, he shows how to calculate the residues. We would like to remark that in the survey [50], the authors study the residues to distribution (not necessarily involutive).

Theorem 2.5. *Let U be a polydisk about the origin 0 in $\mathbb{C}^n$ and let $\mathcal{F} =< \omega >$ be a codimension one holomorphic foliation on U with an isolated singularity at 0. Then we have*

$$\mathrm{Res}_{c_n}(\mathcal{F}; 0) = (-1)^n (n-1)! \dim_{\mathbb{C}} Ext^1_{\mathcal{O}}(\Omega_{\mathcal{F}}, \mathcal{O}) \quad in \quad H_0(0; \mathbb{Q}) = \mathbb{Q}.$$

In the same way to compute residues, F. Bracci and T. Suwa in [10] consider smooth deformations of holomorphic foliations and verify that it provides an effective way to get compute residues.

Theorem 2.6. *Let $(\tilde{M}, \hat{P}, \pi)$ be a deformation of manifolds and $\tilde{\mathcal{F}}$ a deformation of foliations on $\tilde{M}$ of rank p. Suppose that $\mathcal{N}_{\tilde{\mathcal{F}}}$ admits a $\mathcal{C}^\infty$ locally free resolution. Let $S'(\tilde{\mathcal{F}}) \subset S(\tilde{\mathcal{F}})$ be a connect component of the singular set of $\tilde{\mathcal{F}}$ and let $S_t := M_t \cap S'(\tilde{\mathcal{F}})$. Assume that for all $t \in \hat{P}$ the set S_t is compact and $S_t \neq M_t$. Let φ be a homogeneous symmetric polynomial of degree $d > n-p$. Under this assumptions, the Baum-Bott residue $BB_\varphi(\mathcal{F}_t; S_t)$ is continuous in $t \in \hat{P}$. Namely, for any $\mathcal{C}^\infty$ $(2n-2d)$-form $\tilde{\tau}$ on $\tilde{M}$ such that $i_t^*(\tilde{\tau})$ is closed for all $t \in \hat{P}$,*

$$\lim_{t \to t_0} BB_\varphi(\mathcal{F}_t; S_t)(i_t^*(\tilde{\tau})) = BB_\varphi(\mathcal{F}_{t_0}; S_{t_0})(i_{t_0}^*(\tilde{\tau})).$$

It should be noted that for higher dimension foliations it is possible to relate the Baum-Bott residue with the Grothendieck residue. Vishik in [51] founds this relation with the hypothesis that the foliation has locally free tangent sheaf. In [7] Baum and Bott, before and independent of Vishik, have proved a similar result using a generic assumption in the singular set of foliation.

Let us consider $\mathcal{F}$ be a holomorphic foliation of codimension k on a complex manifold M and φ a homogeneous symmetric polynomials of degree $k+1$. Note that $\deg\varphi > n - \dim(\mathcal{F})$, which is condition to Bott vanishing theorem. We consider that the singular set of $\mathcal{F}$ has pure expected codimension, i.e., $\dim(\mathrm{Sing}(\mathcal{F})) = k+1$. In this case it is common to use the notation $\mathrm{Sing}_{k+1}(\mathcal{F})$ to denotes the union of irreducible components of the the singular set of the $\mathcal{F}$ of pure codimension $k+1$.

If $Z \subset \mathrm{Sing}_{k+1}(\mathcal{F})$ is a pure dimension and irreducible component, we consider B_p a $(k+1)$-dimension ball centered at p sufficiently small and transversal to Z at p. We remark that the restricted foliation $\mathcal{F}|_{B_p}$ is an one-dimensional foliation with isolated singularity at p. In [23] (Theorem 1.2), M. Corrêa and F. Lourenço relate the Baum-Bott residue of $\mathcal{F}$ in Z with the Grothendick residue of $\mathcal{F}|_{B_p}$ at p.

Theorem 2.7. *Let $\mathcal{F}$ be a singular holomorphic foliation of codimension k on a compact complex manifold M such that cod*$(\mathrm{Sing}(\mathcal{F})) \geq k+1$. *Then,*

$$\mathrm{Res}_{\varphi}(\mathcal{F}; Z) = \mathrm{Res}_{\varphi}(\mathcal{F}|_{B_p}; p)[Z],$$

where $\mathrm{Res}_{\varphi}(\mathcal{F}|_{B_p}; p)$ *represents the Grothendieck residue at p of the one dimensional foliation $\mathcal{F}|_{B_p}$ on a $(k+1)$-dimensional transversal ball B_p.*

The next result is due to Fernandez-Perez and Tamara in [31] (Theorem 6.2). This provides another effective way to computing Baum-Bott residues of codimension one holomorphic foliations. Before we need of the follow definition

Definition 2.8. We say that the germ $\mathcal{F}$ is an *almost Liouvillian foliation* at $0 \in \mathbb{C}^n$ if there exists a germ of closed meromorphic 1-form γ_0 and a germ of holomorphic 1-form γ_1 at $0 \in \mathbb{C}^n$ such that

$$d\omega = (\gamma_0 + \gamma_1) \wedge \omega.$$

We say that $\mathcal{F}$ is a *simple almost Liouvillian foliation* at $0 \in \mathbb{C}^n$ if we can choose γ_0 having only first-order poles.

Theorem 2.9. *Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^n, n \geq 3$, of a simple almost Liouvillian foliation defined by $\omega \in \Omega^1(\mathbb{C}^n, 0)$ such that*

$$d\omega = (\gamma_0 + \gamma_1) \wedge \omega.$$

Let V be the divisor of poles of $\gamma = \gamma_1 + \gamma_1$ and $V_1, \dots, V_l$ the irreducible components of V. Let Z be an irreducible component of $\mathrm{Sing}_2(\mathcal{F})$*. Then*

$$BB(\mathcal{F}; Z) = \sum_{i=1}^{k} \mathrm{Res}(\gamma_0, V_j) Var(\mathcal{F}, V_j; Z),$$

where $V_1, \dots, V_k$ are the irreducible components that contains Z and $Var(\mathcal{F}, V_j; Z)$ represents the Varational index defined by Khanedani and Suwa in [4].

In [11, 45] was developed the notion of Nash residue of foliations, that immediately implies an aforementioned partial answer to the rationality conjecture of Baum and Bott (see [7], p.287). Let M be a complex manifold of dimension n and $\mathcal{F}$ a singular holomorphic foliation of dimension q on M. Let us consider for each point x in M the following set

$$F(x) := \{v(x); v \in \mathcal{F}_x\}, \tag{2.1}$$

where $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ at x. We observe that $F(x)$ is a subspace of tangent space T_xM of dimension q if, and only if, $x \in M \setminus \mathrm{Sing}(\mathcal{F})$. In general $\dim(F(x)) \leq q$. In the following, $G(q, n)$ is the Grassmannian bundle of q-planes in TM.

Using the express (2.1) we can define a section of $G(q, n)$ outside of singular set of $\mathcal{F}$, as following

$$s : M \setminus \mathrm{Sing}(\mathcal{F}) \longrightarrow G(q, n)$$

gives by $s(x) := F(x)$.

We define $M^\nu := \overline{Im(s)}$ in $G(q,n)$ and call it the Nash modification of M with respect to foliation $\mathcal{F}$. We consider $\mathrm{Sing}(\mathcal{F})^\nu := \pi^{-1}\mathrm{Sing}(\mathcal{F})$ where π is the restriction map to M^ν of the projection of the bundle $G(q,n)$ that is a birational map

$$\pi : M^\nu \longrightarrow M.$$

Moreover, it is biholomorphic from $M^\nu \setminus \mathrm{Sing}(\mathcal{F})^\nu$ to $M \setminus \mathrm{Sing}(\mathcal{F})$. In some case, we can assume M^ν as a smooth manifold (see [45]). We denote by $\tilde{T}^\nu$ and $\tilde{N}^\nu$, respectively, the tautological bundle and the tautological quotient bundle on $G(q,n)$. So, one has a short exact sequence

$$0 \longrightarrow T^\nu \longrightarrow \pi^* TM \longrightarrow N^\nu \longrightarrow 0,$$

where T^ν and N^ν are essentially the restrictions to M^ν.

It is possible to show that the characteristic class $\varphi(N^\nu)$, for a homogeneous symmetrical polynomial φ of degree $d > n - \dim(\mathcal{F})$, is localized at $\mathrm{Sing}(\mathcal{F})^\nu$ give us the following residues

$$\mathrm{Res}_\varphi(N^\nu, \mathcal{F}; \mathrm{Sing}(\mathcal{F})^\nu) \in H_{2(n-d)}(\mathrm{Sing}(\mathcal{F})^\nu; \mathbb{C})$$

and we call it the Nash residue of $\mathcal{F}$ with respect to φ at $\mathrm{Sing}(\mathcal{F})^\nu$.

In 1989, Sertöz in [45] (Theorem IV.4, p.238), has showed that the difference between the Baum-Bott residue and the Nash residue is an integer number with the assumption that M^ν is non singular.

Theorem 2.10. *Let S be a connect component of singular set of $\mathcal{F}$ and φ a homogeneous symmetrical polynomial of degree $d > n - \dim(\mathcal{F})$ then*

$$\mathrm{Res}_\varphi(N_{\mathcal{F}}, \mathcal{F}; S) = \mathrm{Res}_\varphi(N^\nu, \mathcal{F}; S^\nu) + k,$$

where k is a homology cycle in S and it is calculate by a Grassmaann graph construction.

In 2000, Brasselet and Suwa in [11] (Theorem 4.1, p. 44), have given a similar result of Sertöz droping the hypothesis that M^ν is smooth.

Theorem 2.11. *Let φ be a homogeneous symmetric polynomial of degree $d > n - \dim(\mathcal{F})$. If φ is with integral coefficients, then the difference*

$$\mathrm{Res}_\varphi(N_{\mathcal{F}}, \mathcal{F}; S) - \mathrm{Res}_\varphi(N^\nu, \mathcal{F}; S^\nu)$$

is in the image of the canonical homomorphism

$$H_{2(n-p)}(S; \mathbb{Z}) \longrightarrow H_{2(n-p)}(S; \mathbb{C}).$$

In the following result F. Bracci and T. Suwa have developed the residue theory for foliations of adequate singular pairs (see [9]). In short, we consider M a complex manifold of dimension m and let $P \subset M$ be a complex submanifold of dimension r, then we have a short exact sequence

$$0 \longrightarrow TP \longrightarrow TM|_P \longrightarrow N_{P,M} \longrightarrow 0,$$

where $N_{P,M}$ denotes the normal bundle of P in M. We pick X another submanifold of M of dimension n which intersects P along a submanifold $Y \subset M$ of dimension $n+r-m$ and such intersection is everywhere transversal. We define (X, Y) as adequate singular pair in M if $r = m + l - n$ and the data satisfy the following

1) $Y = X \cap P$;
2) $\dim(\mathrm{Sing}(X) \cap P) < l$;
3) X_{reg} intersects P generically transversely.

Theorem 2.12. *([9], Theorem 2.1, p.7) Let (X, Y) be an adequate singular pair in M and let $\mathcal{F}$ be a holomorphic foliation in X of dimension $d \leq l$ which leaves Y invariant. Let $\Sigma = (\mathrm{Sing}(\mathcal{F}) \cup \mathrm{Sing}(Y)) \cap Y$ and assume that $\dim(\Sigma) < l$. Let $\Sigma = \sum_\gamma \Sigma_\gamma$ be the decomposition into connected components and let $i_\gamma : \Sigma_\gamma \hookrightarrow Y$ denotes the inclusion. Let φ be a symmetric homogeneous polynomial of degree $t > l - d$. Then* *i) For each compact connected component Σ_γ there exists a class $\mathrm{Res}_\varphi(\mathcal{F}, Y; \Sigma_\gamma) \in H_{2l-2t}(\Sigma_\gamma; \mathbb{C})$ called "residue", which depends only on the local behavior of $\mathcal{F}$ near Σ_γ;*

ii) If Y is compact we have

$$\sum_\gamma (i_\gamma)_* \mathrm{Res}_\varphi(\mathcal{F}, Y; \Sigma_\gamma) = \varphi(N_{P,M}) \cap [Y] \quad in \quad H_{2l-2t}(Y; \mathbb{C}).$$

We note that this "new concept" of residue of foliations can be understood as a generalization of the classical Camacho-Sad residue theorem (see [16]).

CorrÃªa at al in [29] showed the follows residue formula for orbifolds. Let X be a complex orbifold of dimension n and L be a line V-bundle over X and considering some Chern classes of the bundle $TX - L^\vee$, moreover, for each point p which vanishing ξ, let

$$\pi_p : (\tilde{U}, \tilde{p}) \to (U, p),$$

be a smoothing covering of X at p and the notation $\tilde{\xi} = \pi_p^* \xi$, one has

Theorem 2.13. *([29], Theorem 3.1, p.2897) Let X be a compact orbifold of dimension n with only isolated singularities, let L be a locally V-free sheaf of rank 1 over X and L the associated line V-bundle. Suppose ξ is a holomorphic section of $TX \otimes L$ with isolated zeros. If P is an invariant polynomial of degree n, then*

$$\int_X P(TX - L^\vee) = \sum_{p|\xi(p)=0} \frac{1}{\# G_p} \mathrm{Res}_{\tilde{p}} \Big[P(J\tilde{\xi}) \frac{d\tilde{z}_1 \wedge \ldots \wedge d\tilde{z}_n}{\tilde{\xi}_1 \ldots \tilde{\xi}_n} \Big],$$

where $G_p \subset Gl(n, \mathbb{C})$ denotes a small finite group.

All the results that are well known in residue theory consider the hypothesis that the component of singular set is nondegenerate, see for instance [7]. The paper [20] provides a slight improvement of the results given by Baum-Bott by considering the degenerate case with restrictions. Let us consider v a holomorphic vector field on V and W a component of singular set of v such that the vector field is degenerate along of W. M. Dia proves the result of residue of v at W subject to the condition that there is a biholomorphism between a neighborhood of W and a neighborhood of the zero section of the normal bundle of W. See Théorème A, B, C and D in [20].

2.4 Residues of logarithmic foliations of dimension one

This section is dedicated to show some results about residues logarithmic associates to holomorphic foliations of dimension one. For this we refer to readers [2, 24, 21, 34, 38] and the references therein.

The general index of a vector field tangent to hypersurfaces was defined and studied in terms of the homology of the complex of differential forms by X. Gomez-Mont, L. Giraldo and P. Mardesić, see [34, 38]. The first result in this section is due A. G. Aleksandrov in [2] which is about the logarithmic index. The author defines a logarithmic index using differential forms with logarithmic poles and determines the relation with the homological index.

Let M be a complex manifold of dimension n, and let $\Omega^q_M, q \geq 0$, and $Der(M)$ be the sheaves of germs of holomorphic q-forms and vector fields on M, respectively. Let $D \subset M$ be a divisor which all of whose irreducible components are of multiplicity one. Given $V \in Der(M)$ a vector field which has an isolated singularity at a point $x \in D$ then the ι_V-homology groups of the complex $(\Omega^{\bullet}_{D,x}, \iota_V)$ are finite-dimensional vector spaces, where $\iota_V : \Omega^q_M \to \Omega^{q-1}_M$ is the interior multiplication (contraction). We can define the homological index of the vector field V at the point $x \in D$ by

$$Ind_{hom,D,x}(V) := \sum_{i=0}^{n} (-1)^i \dim H_i(\Omega_{D,x}, \iota_V).$$

To talk about logarithmic index, given a divisor D we can consider the coherent analytic sheaves $\Omega^q_M(\log D), q > 0$ and $Der_M(\log D) = T_M(-\log D)$

as in [2]. Consider a vector field $V \in Der_M(\log D)$. As above, the interior multiplication ι_V defines the structure of a complex on $\Omega^\bullet_M(\log D)$.

Lemma 2.14. *([2], Lemma 1, p.247) If all singularities of V are isolated, then the ι_V-homology groups of the complex $\Omega^\bullet_M(\log D)$ are finite-dimensional vector spaces.*

With this preliminary result is well defined the *logarithmic index* of the field V at the point x,

$$Ind_{\log D,x}(V) := \sum_{i=0}^{n} (-1)^i \dim H_i(\Omega_{D,x}(\log D), \iota_V).$$

These index are related bellow.

Proposition 2.15. *([2], Proposition 1, p. 248) Suppose that a point $x \in D$ is an isolated singularity of a vector field $V \in Der(\log D)$, the germs $V_i \in \mathcal{O}_{M,x}$ are determined by the expansion $V = \sum_i V_i \dfrac{\partial}{\partial z_i}$, and $J_x V = (V_0, \ldots, V_n)\mathcal{O}_{M,x}$. Then*

$$Ind_{hom,D,x}(V) = \dim \mathcal{O}_{M,x}/J_x V - Ind_{\log D,x}(V).$$

In [24] the authors consider logarithmic foliation along D and prove the residue formulas, namely, Baum-Bott type formulas for non-compact complex manifold, still considering the logarithmic vector field.

Theorem 2.16. *([24], Theorem 1, p. 6403) Let $\tilde{X}$ be an n-dimensional complex manifold such that $\tilde{X} = X - D$, where X is an n-dimensional complex compact manifold and D is a smooth hypersurface on on X. Let $\mathcal{F}$ be a foliation of dimension one on X with isolated singularities and logarithmic along D. Suppose that $Ind_{\log D,p}(\mathcal{F}) = 0$ for all $p \in \mathrm{Sing}(\mathcal{F}) \cap D$. Then*

$$\int_X c_n(T_X(-\log D) - T_{\mathcal{F}}) = \sum_{p \in \mathrm{Sing}(\mathcal{F}) \cap (X \setminus D)} \mu_p(\mathcal{F}).$$

In the same work, the authors consider that the divisor D is a normal crossing hypersurface and one has

Theorem 2.17. *([24], Theorem 2, p. 6404) Let $\tilde{X}$ be an n-dimensional complex manifold such that $\tilde{X} = X - D$, where X is an n-dimensional complex compact manifold, D is a normally crossing hypersurface on X. Let $\mathcal{F}$ be a foliation on X of dimension one, with isolated singularities (non-degenerates) and logarithmic along D. Then,*

$$\int_X c_n(T_X(-\log D) - T_{\mathcal{F}}) = \sum_{p \in \mathrm{Sing}(\mathcal{F}) \cap (\tilde{X})} \mu_p(\mathcal{F}).$$

In [21] the authors prove new versions of Gauss-Bonnet and PoincarÃ©-Hopf theorems for complex ∂-manifolds of the type $\tilde{X} = X - D$, where $\dim X = n \geq 3$ and D is a reduced divisor. More precisely,

Theorem 2.18. *([21], Theorem 1.1 p. 495) Let $\tilde{X}$ be a complex manifold such that $\tilde{X} = X - D$, where X is an n-dimensional ($n \geq 3$) complex compact manifold and D is a reduced divisor on X. Given any (not necessarily irreducible) decomposition $D = D_1 \cup D_2$, where D_1, D_2 have isolated singularities and $C = D_1 \cap D_2$ is a codimension 2 variety and has isolated singularities,*

(i) (Gauss-Bonnet type formula) the following formula holds

$$\int_X c_n(\Omega^1_X(\log D)) = (-1)^n \chi(\tilde{X}) + \mu(D_1, S(D_1)) + \mu(D_2, S(D_2)) - \mu(C, S(C)).$$

(ii) (Poincaré-Hopf type formula) if v is a holomorphic vector field on X, with isolated singularities and logarithmic along D, we have that

$$\chi(\tilde{X}) = PH(v, \mathrm{Sing}(v)) - GSV(v, D_1, S(v, D_1)) - GSV(v, D_2, S(v, D_2)) +$$
$$+ GSV(v, C, S(v, C)) + (-1)^{n-1} \left[\mu(D_1, S(D_1)) + \mu(D_2, S(D_2)) - \mu(C, S(C))\right].$$

2.5 Residues to flags

In the following, we review some results about residues of flag of foliations which have emerged in recent years. For the background of flags, we refer to [37, 41, 41, 27] and the references therein.

The next result ([37], Theorem 2) has a twofold proposal. The first one it is shows that given a flag $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ of holomorphic foliations on M we can talk about residues with an element of homology group, and the second one, under the certain condition in M, one has the Baum-Bott type theorem.

Theorem 2.19. *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a 2-flag of holomorphic foliations on a compact complex manifold M of dimension n. Let φ_1, φ_2 be homogeneous symmetric polynomials, respectively of degrees d_1 and d_2, satisfying the Bott vanishing theorem to Flags. Then for each compact connected component S of $\mathrm{Sing}(\mathcal{F})$ there exists a class, $\mathrm{Res}_{\varphi_1,\varphi_2}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}; S) \in H_{2n-2(d_1+d_2)}(S;\mathbb{C})$, that we will call it of Baum-Bott Reisdue of Flag, such that*

$$\sum_{\lambda} (\iota_\lambda)_* \mathrm{Res}_{\varphi_1,\varphi_2}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}; S_\lambda) = (\varphi_1(\mathcal{N}_{12}).\varphi_2(\mathcal{N}_2)) \frown [M] \tag{2.2}$$

in $H_{2n-2(d_1+d_2)}(M;\mathbb{C})$, where ι_λ denotes the embedding of S_λ in M.

To approach of a good expression to residues, we show how the residues of foliations, in a flag, are related (see [37], Proposition 3, p. 1169).

Theorem 2.20. *For a flag $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ on M with $\dim(\mathcal{F}_1) = \mathrm{codim}(\mathcal{F}_2) = 1$ and $\mathrm{Sing}(\mathcal{F}_1) \cap \mathrm{Sing}(\mathcal{F}_2)$ admitting isolated singularities (only) we have*

$$\mathrm{Res}_{c_n}(\mathcal{F}_2, \mathcal{N}_2; p) = (-1)^n (n-1)! \mathrm{Res}_{c_n}(\mathcal{F}_1, \mathcal{N}_1; p),$$

where the residues involved are of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$.

To continue into the goal of to show expression to residue, we consider $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ a flag on M with notation $\mathrm{codim}(\mathcal{F}_i) = k_i$ to $i = 1, 2$. Let us consider the notation that $\mathrm{Sing}_{k_i+1}\mathcal{F}_i$ represents the union of irreducible components of $\mathrm{Sing}(\mathcal{F}_i)$ of pure codimension k_i+1.

Let us take an irreducible component $Z \subset \mathrm{Sing}_{k_1+1}(\mathcal{F}_1)$ and a generic point $p \in Z$. So, we pick B_p a small ball centered at p such that $S(B_p) \subset B_p$ is a sub-ball of dimension $n-k_1-1$(same dimension than the component Z). Thus, de Rham class can be integrated over an oriented $(2k_1+1)$-sphere $L_p \subset B_p^*$ and one has the notation

$$BB^j(\mathcal{F};Z) := (2\pi i)^{-k_1-1}\int_{L_p} \theta^{12}\wedge(d\theta^2)^j\wedge(d\theta^{12})^{k_1-j} \;\text{ for each }\; 0 \leq j \leq k_2.$$

For this particular flag, we get the formula to residue and the Baum-Bott theorem, see ([37], Theorem 4, p.1173).

Theorem 2.21. *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a 2-flag of codimension (k_1, k_2) on a compact complex manifold M. If* $\mathrm{codim}(\mathrm{Sing}(\mathcal{F})) \geq k_1+1$*, then for each $0 \leq j \leq k_2$ we have*

$$c_1^{k_1-j+1}(\mathcal{N}_{12}) \smile c_1^j(\mathcal{N}_2) = \sum_{Z\,\in\,\mathrm{Sing}_{k_1+1}(\mathcal{F}_1)\cup\mathrm{Sing}_{k_1+1}(\mathcal{F}_2)} \lambda_Z^j(\mathcal{F})[Z],$$

where $\lambda_Z^j(\mathcal{F}) = BB^j(\mathcal{F}, Z)$.

In 2020 Ferreira and Lourenço in [32] extend the residue theory to flag of holomorphic distributions and prove some results of this residues. The next result is relates about the isolates singularities.

Theorem 2.22. *([32], Theorem 1.2) Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a 2-flag of holomorphic distributions on a compact complex manifold M of dimension n, φ_1 and φ_2 be homogeneous symmetric polynomials, respectively of degrees $d_1 > 0$ and $d_2 > 0$ and p an isolated point of* $\mathrm{Sing}(\mathcal{F})$*. Then*

$$\mathrm{Res}_{\varphi_1,\varphi_2}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}; p) = 0.$$

When we consider a particular manifold as the projective space $M = \mathbb{P}^3$ we obtain some advances in goal of calculate residue of flags. It follows an effective way to calculate residue when the singular scheme of flag has only one irreducible component, see ([32], Theorem 1.3).

Theorem 2.23. *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a 2-flag of holomorphic foliations on $\mathbb{P}^3$ with $\deg(\mathcal{F}_i) = d_i$ thus*

$$(1 + d_1 - d_2) \sum_{Z \in S_1(\mathcal{F}_2)} \deg(Z)\mathrm{Res}_{\varphi_2}(\mathcal{F}_2|_{B_p}; p) = \sum_{Z \in S_1(\mathcal{F})} \mathrm{Res}_{c_1\varphi_2}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}; Z), \tag{2.3}$$

where $\deg(Z)$ is the degree of the irreducible component Z, $\mathrm{Res}_{\varphi_2}(\mathcal{F}_2|_{B_p}; p)$ represents the Grothendieck residue of the foliation $\mathcal{F}_2|_{B_p}$ at $\{p\} = Z \cap B_p$ with B_p a transversal ball and either $\varphi_2 = c_1^2$ or $\varphi_2 = c_2$.

We believe that if we work with residue currents, see [40], meanly in degenerate case, we must give an express more general to calculate residues.

3 Some applications

This section is dedicates to show how to use the residue theory in open problems.

3.1 Residues and the Poincaré problem

In 1891 Henri Poincaré, studying the algebraic integrability of equations and motivated by Darboux's works, to appear the following question, see [44]

"Is it possible to bound the degree of an irreducible curve such that is invariant by a foliation in terms of the degree of foliation?"

This problem is similar to decide whether a holomorphic foliation on $\mathbb{P}^2$ admits a rational first integral. This question is known as the *Poincaré problem.* Although it is well known that such a bound does not exist in general, see [19], under certain hypotheses, there are several works about this problem which answer it partially and there are several generalizations even for flags and for Pfaff systems; see for instance [14, 17, 19, 30, 33, 43, 46, 26, 27, 32, 25]. The residue theory, in special the Baum-Bott Theorem, are powerful tool and obstructions of several problems related to foliations with singularities.

After 100 years, in ([19], Theorem 1, p.891) Cerveau and Lins Neto given a first partial answer to Poincaré problem. In order to present it consider S a projective nodal curve, that is all its singularities are of normal crossing type, with reduced homogeneous equation $f = 0$, of degree m.

Theorem 3.1. *Let $\mathcal{F}$ be a foliation in $\mathbb{CP}(2)$ of degree n, having S as separatrix. Then $m \leq n+2$. Moreover if $m = n+2$ then f is reducible and $\mathcal{F}$ is of logarithmic type, that is given by a rational closed form $\sum \lambda_i \dfrac{df_i}{f_i}$ where $\lambda_i \in \mathbb{C}$ and the f_i are homogeneous polynomials.*

Some years later, Carnicer in ([17], Theorem, p.289) presents the same quota to Poincaré problem with other hypotheses, namely, $\mathcal{F}$ does not have dicritical singularities (i.e., singularity with infinitely many invariant curves passing through it) into the curve.

Theorem 3.2. *Let $\mathcal{F}$ be a foliation of $\mathbb{P}^2$ and let C be an algebraic curve in $\mathbb{P}^2$. Suppose that C is invariant by $\mathcal{F}$ and there are no dicritical singularities of $\mathcal{F}$ in C. Then*

$$\deg C \leq \deg \mathcal{F} + 2.$$

M. G. Soares in ([46], Theorem B, p.496) extends the Poincaré problem from $\mathbb{P}^2$ to $\mathbb{P}^n$ improving the above quota with another hypotheses and proved the result by applying Baum-Bott Theorem. Let $V \hookrightarrow \mathbb{P}^n, n \geq 2$, be an irreducible non-singular algebraic hypersurface of degree d_0 invariant by $\mathcal{F}^d$, a non-degenerated one-dimensional holomorphic foliation of degree $d \geq 2$. Then we have.

Theorem 3.3.

$$d_0 \leq d + 1.$$

In [12] Brunella says that the GSV-index is the obstruction to a positive solution to Poincaré problem and gives a simple condition that implies the non negativity of this index. Motivated by Brunella's work, Corrêa and Machado in [25] introduce a GSV type index for invariant varieties by holomorphic Pfaff systems on projective manifolds. The authors prove, with certain hypotheses, a non negativity property for this index. As a consequence one has the result.

Theorem 3.4. *Let $\omega \in H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d + k + 1))$ be a holomorphic Pfaff system of rank k and degree d. Let $V \subset \mathbb{P}^n$ be a reduced complete intersection variety, of codimension k and multidegree $(d_1, \ldots, d_k)$, invariant by ω. Suppose that $\operatorname{Sing}(\omega, V)$ has codimension one in V, then*

$$\sum_i GSV(\omega, V, S_i) \deg(S_i) = [d + k + 1 - (d_1 + \cdots + d_k)](d_1 \ldots d_k),$$

where S_i denotes an irreducible component of $\operatorname{Sing}(\omega, V)$. In particular, if $GSV(\omega, V, S_i) \geq 0$, for all i, we have

$$d_1 + \cdots + d_k \leq d + k + 1.$$

3.2 Residues and the non-existence of minimal sets

The residue theory can be used in problem of existence or not of non-trivial minimal sets for foliations, as we list some works in the sequence.

The idea of the minimal set was firstly considered by Camacho, Lins Neto and Sad in [17], with the main objective to understand more geometrical information about the foliations.

Let $\mathcal{F}$ be a holomorphic foliation on a compact manifold X. A compact non-empty subset $\mathcal{M} \subset X$ is said to be a *non-empty minimal set* for $\mathcal{F}$ if the following properties are satisfied

a) $\mathcal{M}$ is invariant by $\mathcal{F}$;

b) $\mathcal{M} \cap \mathrm{Sing}(\mathcal{F}) = \emptyset$;

c) $\mathcal{M}$ is minimal with respect to properties *a*) and *b*).

In [17] the authors addresses the problem of the existence or not of non-trivial minimal set for codimension one foliations of $\mathbb{P}^2$. More precisely, they prove a geometrical property of minimal sets, i.e., by applying the Maximum Principle for harmonic functions, they prove that $\mathcal{F}$ has at most one non-trivial minimal set. Moreover, under generic conditions imposed on the singularities of foliation, all leaves accumulate on that set. Anyway, in general, the question of existence of non-trivial minimal sets for holomorphic foliations on $\mathbb{P}^2$ remains as an open problem.

In ([42], Theorem 1, p. 1370) Lins Neto studies the problem of the existence of non-trivial minimal set of codimension one holomorphic foliation on $\mathbb{P}^n, n \geq 3$ and he proves the following result

Theorem 3.5. *Codimension 1 foliations on $\mathbb{P}^n, n \geq 3$, have no non-trivial minimal sets.*

In ([15], Theorem 1.2, p.296] the authors, by using Baum-Bott Theorem, have generalized Lins Neto's result for codimension one holomorphic foliations on projective manifolds with cyclic Picard group.

Theorem 3.6. *Let X be a complex projective manifold of dimension at least 3 and with $Pic(X) = \mathbb{Z}$, and let $\mathcal{F}$ be a codimension one foliation on X. Then every leaf L of $\mathcal{F}$ accumulates to $\mathrm{Sing}(\mathcal{F})$:*

$$\overline{L} \cap \mathrm{Sing}(\mathcal{F}) \neq \emptyset.$$

The Theorem 3.6 is a partial answer to Brunella's conjecture, see ([13], Conjecture 1.1, p.3102), in the special case when $\mathrm{Pic}(X) = \mathbb{Z}$. Recently, in 2021 M. Adachi and J. Brinkschulte have proved the Brunella's conjecture without hypotheses on manifold X , see ([1], Main Theorem, p.1).

Theorem 3.7. *Let X be a compact complex manifold of dimension ≥ 3. Let $\mathcal{F}$ be a codimension one holomorphic foliation on X with ample normal bundle $N_{\mathcal{F}}$. Then every leaf of $\mathcal{F}$ accumulates to* $\mathrm{Sing}(\mathcal{F})$.

For higher codimensional foliations, Brunella's conjecture has been stated ([22], Conjecture 1.2, p. 1236) and still remains as an open problem. However, in [28] the authors by using a Brunella-Khanedani-Suwa variational type residue theorem for currents invariant by holomorphic foliations, provide conditions for the accumulation of the leaves to the intersection of the singular set of a holomorphic foliation with the support of an invariant current.

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