UNIVERSIDADE FEDERAL DE LAVRAS

RAFAEL MANCINI SANTOS

ON THE UNRUH EFFECT IN AN EFFECTIVE KLEINIAN GEOMETRY

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Dissertação apresentada à Universidade Federal de Lavras, como parte das exigências do Programa de Pós-Graduação em Física, área de concentração em Física, para a obtenção do título de Mestre.

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To see a World in a grain of sand,
And a Heaven in a wild flower,
Hold Infinity in the palm of your hand,
And Eternity in an hour
(William Blake)

RESUMO

A verificação experimental direta do efeito Unruh tem evadido físicos por muito tempo. Entretanto, esse cenário pode estar prestes a mudar. Sistemas análogos de metamateriais podem providenciar alternativas promissoras para a observação do efeito Unruh. Ademais, fótons, em alguns desses sistemas, comportam-se como se estivessem em um espaço equipado com uma assinatura Kleiniana $(++--)$ — um *espaço Kleiniano*. Neste trabalho, o efeito Unruh em uma geometria efetiva com assinatura Kleiniana oriunda de um meio anisotrópico, como um metamaterial hiperbólico, é investigado. A abordagem por integrais de trajetória, para o efeito Unruh, é aplicada no contexto de um espaço Kleiniano, em que surgem problemas. É destacado que qualquer sistema análogo de metamateriais pode estar, na verdade, mimetizando um fenômeno físico conhecido em um espaço-tempo exótico onde propriedades já estabelecidas se tornam inválidas. No caso deste trabalho, um referencial acelerado análogo no espaço Kleiniano é derivado e uma compactificação induzida de uma dimensão tipo-tempo, assim como a ausência de "cunhas", em contraste com o que ocorre no espaço de Rindler, são observadas. Apesar do efeito Unruh análogo no espaço Kleiniano não ser demonstrado, algumas perspectivas e alternativas são discutidas.

Palavras-chave: Efeito Unruh. Espaço Kleiniano. Geometria não Lorentziana.

ABSTRACT

Direct experimental verification of the Unruh effect has eluded physicists for a long time. However, this scenario may be changing. Metamaterial analog systems may provide promising alternatives for the observation of the Unruh effect. Furthermore, photons, in some of these systems, behave as if they were in a spacetime endowed with a Kleinian signature $(++--)$ — a *Kleinian space*. In this work, the Unruh effect in an effective geometry with a Kleinian signature arising from an anisotropic medium, such as a hyperbolic metamaterial, is investigated. The path integral approach to the Unruh effect is applied in the context of a background space with Kleinian signature, where problems arise. It is noted that any metamaterial analog system might be, in fact, mimicking a known physical phenomenon but in an exotic space-time where some established physical properties may break down. In the case of this work, an analog accelerated frame in Kleinian space is derived and an induced compactification of a timelike dimension as well as the absence of “wedges”, in contrast to what occurs in Rindler space, are observed. Although an analog Unruh effect in Kleinian space is not shown, some perspectives and alternatives are discussed.

Keywords: Unruh effect. Kleinian space. Non-Lorentzian geometry.

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1 INTRODUCTION

The Unruh effect is a direct prediction of Quantum Field Theory (Unruh, 1976), and therefore it can be interpreted as a natural consequence of Quantum Mechanics and Special Relativity. It states that uniformly accelerating observers experience the Minkowski vacuum as a thermal bath with temperature $T = \hbar a/2\pi ck$, where a is the observer's acceleration, $\hbar$ is the reduced Planck constant, c is the speed of light, and k is the Boltzmann's constant. Since an acceleration of 10^{20} m/s^2 is necessary to achieve a temperature of the order of 1K, its direct experimental verification is still a difficult challenge.

Despite this, there are good reasons to believe that it is possible to detect the Unruh effect using quantum systems (Martín-Martínez; Fuentes; Mann, 2011; Martín-Martínez et al., 2013) as well as classical ones (Cozzella et al., 2017), with experiments that may be accessible with the current technology. Indeed, there is a report of the observation of thermalization at the Unruh temperature caused by radiation emitted from accelerated positrons, which may be the first direct measure of the Unruh temperature in a high-energy system (Lynch et al., 2021).

One possible route to overcome the necessary acceleration to detect the Unruh effect is through metamaterial waveguides, in which photons behave as massive quasiparticles. They can achieve accelerations as high as $10^{24}g$, where g is the Earth's gravitational acceleration, due to the anisotropic character of the metamaterial waveguide. (Smolyaninov, 2019).

These metamaterials are artificially designed materials with exotic properties that stem both from its structure and chemical composition (Cui; Smith; Liu, 2010). Although they are not a novelty, having a history that dates back to the time of the Romans (Leonhardt, 2007), modern metamaterials offer an unprecedented degree of control over its electronic properties, i.e., their permittivity and magnetic permeability tensors — ε_{ij} and μ_{ij}, respectively (Smolyaninov, 2018).

This degree of control allows designs of electromagnetic metamaterials which act as effective geometries, enabling one to construct metamaterial analog systems capable of modelling spacetimes (Leonhardt; Philbin, 2006). In this sense, it has been observed that light propagation in some electromagnetic metamaterials experience a metric with the signature $(--++)$ — a *Kleinian signature* (Smolyaninov; Narimanov, 2010).

Investigations concerning the Kleinian signature goes back to 1994, with the works of Alty (1994), where fields propagating through a wall of signature change from Lorentzian to Kleinian were investigated, and of Barrett et al. (1994), where the Kleinian geometry arising from its signature was studied in the context of self-dual gravity. This kind of geometry is also present in investigations concerning the "two-time physics" (Bars, 2000; Foster; Müller, 2010).

In some metamaterials, light experiences a metric with a Kleinian signature when its frequency is low, and a metric with a Lorentzian signature when its frequency is high (Smolyaninov, 2007). This phenomenon enables the investigation of metric phase transitions, as discussed by Sakharov (1984), to be done as a function of light frequency in this analog setting. Moreover, it enables one to investigate well known phenomena in a setting where established properties might break down.

As an example, in an anisotropic metamaterial with dielectric permittivities $\varepsilon_1 = \varepsilon_x = \varepsilon_y$ and $\varepsilon_2 = \varepsilon_z$, the equation of motion for the extraordinary wave is (Smolyaninov, 2007)

$$\frac{\partial^2 E}{c^2 \partial t^2} = \frac{\partial^2 E}{\varepsilon_1 \partial z^2} + \frac{\partial^2 E}{\varepsilon_2 \partial x^2} + \frac{\partial^2 E}{\varepsilon_2 \partial y^2}. \tag{1.1}$$

In some indefinite metamaterials, i.e. metamaterials in which not all of the principal components of the permittivity tensors have the same sign (Smith; Schurig, 2003), it can happen that $\varepsilon_1 < 0$ while $\varepsilon_2 > 0$. Since one can write Eq. 1.1 as $g^{\mu\nu}\partial_\mu \partial_\nu E = 0$, where $g_{\mu\nu}$ is the metric of spacetime, one may interpret this as a wave equation in an effective geometry of Kleinian signature.

The proposal by Smolyaninov (2019) relies on electromagnetic theory to show the "giant" Unruh effect. But the Unruh effect is a consequence of quantum field theory. Therefore, in this work, we attempt to provide a derivation of the Unruh effect in an effective geometry with Kleinian signature through the path integral approach, following closely the proposal of Unruh and Weiss (1984), in order to contribute to bridging this gap between the proposed giant Unruh effect in an anisotropic metamaterial medium and its quantum field theory origins, albeit being described in an effective geometry.

For this purpose, we begin by reviewing, in chapter 2, the path integral approach in quantum mechanics, how one develops it from the interpretation of the transition amplitude as a sum of paths, and its applications in field theory, to obtain the vacuum Green's functions, and in finite temperature field theory, where we discuss how one can use the path integral approach to model thermal states.

Then, in chapter 3, we present a derivation of the Unruh effect, through the path integral approach, following the work of Unruh and Weiss (1984) using a real, massive, free scalar field. To do that, we first obtain the generating functional for such system, derive the frame of an accelerated observer following the work of Rindler (1966) and obtain, in this frame, the partition function of the system.

In chapter 4, we partly review the work of Alves-Júnior, Barreto and Moraes (2021) on the implications of relativity in a space with a Kleinian geometry. We discuss how phenomena such as the "time dilation" and "length contraction" is affected in such geometry, how particles behave in this new setting and we present a derivation of an analog accelerated frame.

In chapter 5, we develop an attempt at a derivation of the Unruh effect in an effective Kleinian geometry originating from an anisotropic media, such as an electronic metamaterial. We begin by commenting on how effective geometries emerge from such media. Then, we proceed to apply the method presented in chapter 3, using the analog accelerated frame presented in chapter 4, to calculate the analog Unruh temperature in this context.

Finally, in chapter 6, we conclude our work with some final remarks. We briefly mention some alternative approaches to the problem stated in this work and discuss some of the problems encountered.

2 PATH INTEGRAL QUANTIZATION

> There is a pleasure in recognizing old things from a new point of view (Feynman, 1948, p.71)

In 1948, Richard P. Feynman published his seminal paper introducing to the world a new way to formulate quantum mechanics: the path integral approach (Feynman; Leighton; Sands, 1965). Although it is central to the discussion of gauge theories and QCD nowadays, it was ignored for some time after its proposal partly due to its mathematical issues (Grosche, 1993). In this chapter, we provide an introduction to the Feynman path integral formulation of quantum mechanics and some of its applications to quantum field theory.

Section 2.1 is a review of the propagator in quantum mechanics, its role in quantum dynamics and how it can be interpreted as a sum of all paths. In section 2.2, we introduce the Feynman path integral and discuss briefly some of its issues. Section 2.3 is a brief demonstration of how one can express the Feynman path integral in the Hamiltonian formalism, and it also serves as a preparation for its use in field theory. In section 2.4, we discuss the application of the path integral formulation to field theory. In this context, the path integral goes by the name of *generating functional*. Finally, in section 2.5, we discuss the application of path integrals to field theories at a finite temperature.

2.1 The propagator in quantum mechanics

In quantum mechanics, the time evolution of a quantum state in the absence of measurement is given by a unitary operator known as the *time evolution operator* (Sakurai; Commins, 1995), which we will denote here as $U(t,t_0)$. These operators form a unitary, one-parameter, Abelian group and, by the Stone's theorem, this group is generated by a self-adjoint operator, which we will denote here as H, and its elements can be written as (Teschl, 2009)

$$U(t,t_0) = \exp[-iH(t-t_0)], \tag{2.1}$$

where the self-adjoint operator H is also called the *energy observable*, or *Hamiltonian operator*.

Let $|\psi(t_0)\rangle$ be a one-particle state prepared at time t_0. The action of the time evolution operator $U(t,t_0)$ on $|\psi(t_0)\rangle$ gives the state evolved to time t,

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle. \tag{2.2}$$

Now, let Q be the position operator and $|q\rangle$ be an eigenstate of Q with eigenvalue q. We can write the initial state as a position eigenfunction,

$$\psi(q_0,t_0) = \langle q_0|\psi(t_0)\rangle, \tag{2.3}$$

and rewrite eq. 2.2 as

$$\psi(q,t) = \langle q|U(t,t_0)|\psi(t_0)\rangle. \tag{2.4}$$

It is a common trick[1] to consider the position eigenstates as a basis for the Hilbert space of states such that

$$\mathrm{id} = \int dq|q\rangle\langle q|, \tag{2.5}$$

where "id" denotes the identity operator. Hence, eq. 2.4 can be rewritten as

$$\begin{aligned}\psi(q,t) &= \int dq_0\langle q|U(t,t_0)|q_0\rangle\psi(q_0,t_0)\\ &= \int dq_0 K(q,t;q_0,t_0)\psi(q_0,t_0)\end{aligned} \tag{2.6}$$

where $K(q,t;q_0,t_0)$, the kernel of this integral operator that takes $\psi(q_0,t_0)$ to $\psi(q,t)$, is called the *propagator*, or "Green's function". Note that for a given initial value $\psi(q_0,t_0)$, the knowledge of the propagator $K(q,t;q_0,t_0)$ provides the unitary evolution of the state. In that sense, it is convenient to introduce an energy eigenbasis $\{|e\rangle\}$, with $H|e\rangle = e|e\rangle$, such that the propagator can be expressed as

$$K(q,t;q_0,t_0) = \sum_e \langle q|e\rangle\langle e|q_0\rangle \exp[-ie(t-t_0)]. \tag{2.7}$$

We can separate the exponential to obtain position eigenstates in the Schrödinger picture,

$$K(q,t;q_0,t_0) = \sum_e \langle q|\exp(-iHt)|e\rangle\langle e|\exp(iHt_0)|q_0\rangle \tag{2.8}$$

$$= \langle q,t|q_0,t_0\rangle, \tag{2.9}$$

[1] To work with the position eigenstates in the way it is usually done in Dirac notation, one must equip the Hilbert space of quantum states with a theory of distributions, giving raise to a Rigged Hilbert space (Madrid, 2005)

where, as we can see, the propagator is expressed as the transition amplitude for a state initially prepared at time t_0 on position q_0 to be measured on q at later time t. Since all this is being presented as a preparation for the introduction of the path integral, one can loosely interpret this last expression as the transition amplitude for a particle to go from (q_0, t_0) to (q, t) in spacetime.

Again, it is safe to consider the set $\{|q,t\rangle\}$ as a basis for the Hilbert space, such that

$$\mathrm{id} = \int dq |q,t\rangle\langle q,t|. \tag{2.10}$$

Now, if we partition the interval $[t_0, t] \in \mathbb{R}$ into $[t_0, t']$ and $[t', t]$, we can use 2.10 to write

$$\begin{aligned}\langle q,t|q_0,t_0\rangle &= \int dq' \langle q,t|q',t'\rangle\langle q',t'|q_0,t_0\rangle \\ &= \int dq' K(q,t;q',t')K(q',t';q_0,t_0).\end{aligned}$$

Following this line of reasoning, we can partition the interval (t_0, t) into N small intervals $[t_{i-1}, t_i]$, $i = 1, 2, ..., N$, of Lebesgue measure[2] ε, i.e., $t_i - t_{i-1} = \varepsilon$, such that $t - t_0 = \sum_{i=1}^{N}(t_i - t_{i-1}) = N\varepsilon$. In that way, we can write 2.9 as

$$K(q,t;q_0,t_0) = \int \prod_{i=1}^{N-1} dq_i \prod_{j=1}^{N} K(q_j, t_j; q_{j-1}, t_{j-1}), \tag{2.11}$$

where $(q,t) = (q_N, t_N)$. This result can be interpreted as the transition amplitude from state $|q_0, t_0\rangle$ to $|q, t\rangle$ being given by the sum of all paths with fixed endpoints (q,t) and (q_0, t_0). Figure 2.1 illustrates such paths in the xt plane.

2.2 Feynman path integrals

The Hamiltonian formulation served as the starting point for the design of modern quantum mechanics in the beginning of the 20th century, specially considering Schrödinger's wave mechanics. There is a well known correspondence between geometrical optics and Hamilton's mechanics[3] and it has been said, indeed,

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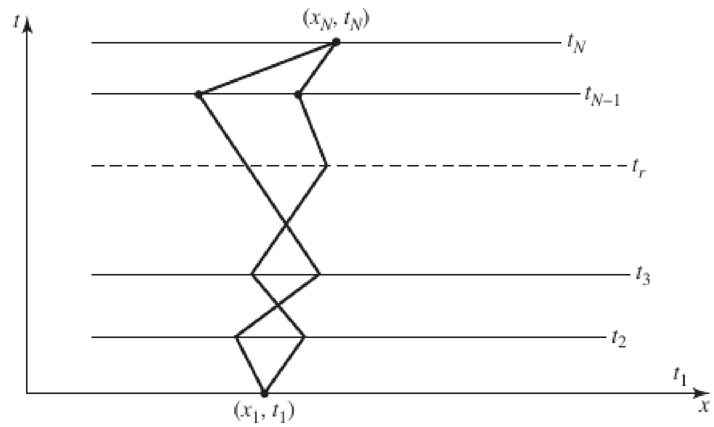
[2] The Lebesgue measure is a map μ that, loosely speaking, assigns "volumes" to subsets of $\mathbb{R}^n$, in general. For an interval $[a,b] \in \mathbb{R}$, $\mu([a,b]) = b - a$. For the interested reader, more information about measures and measure spaces can be found in (Rudin, 1970).

[3] Hamilton's mechanics can be constructed as the geometric optics of higher dimensional space (Arnold, 2013).

Figure 2.1 – Examples of paths in the *xt* plane



Source: (Sakurai; Commins, 1995)

that Hamilton formulated his basic principles of mechanics from his research into optics in non-homogeneous media (Schrödinger, 1926). Furthermore, the Hamilton-Jacobi equation can be seen as establishing a relation between the trajectories of mechanical systems and wave fronts (Arnold, 2013). Schrödinger himself claimed, in one of his seminal works introducing to the world his wave mechanics, that his motivations backing his theory were the aforementioned relation established by the Hamilton-Jacobi equation and L. de Broglie's research on "phase-waves" (Schrödinger, 1926).

However, as quantum theory developed over the course of the 20th and 21st century, the Lagrangian mechanics started to play an increasingly important role, mainly regarding quantum field theory and its developments through a pursuit of nature's symmetries[4]. Indeed, it is easier to design self-consistent quantum field theories through the requirement that its Lagrangian satisfy certain symmetries, since these are related to concepts like the mass, spin and charge of a particle. Furthermore, symmetries are directly related to the fundamental forces of the standard model (Schwichtenberg, 2015).

The Lagrangian mechanics is also central to the discussion of the path integral formalism, which, albeit it is an invaluable tool for the treatment of gauge theories, it already shows its power in quantum

[4] This is a topic subject to debate. Arguments more in favor of the aforementioned "pursuit of nature's symmetries" may be found in Tegmark's paper entitled "The Mathematical Universe" (Tegmark, 2008), while arguments against this can be found in Hossenfelder's book entitled "Lost in Math" (Hossenfelder, 2018)

mechanics. In his famous book *The Principles of Quantum Mechanics*, Dirac stated that (Dirac et al., 1981)

$$\exp\left\{ i\int_{t_a}^{t_b} \frac{L(x,\dot{x})}{\hbar} dt \right\} \text{ corresponds to } \langle q_b, t_b | q_a, t_a \rangle, \tag{2.12}$$

where L is the classical Lagrangian. Feynman, while working on the problem of obtaining the classical limit from quantum mechanics, saw this statement and became puzzled by the meaning of the term "corresponds to", since it wasn't clear from context if this was supposed to mean "equal to" or "proportional to" (Sakurai; Commins, 1995). This puzzling suggestion from Dirac was one of the main motivations for Feynman's proposal of the path integral formulation to quantum mechanics, which, albeit mathematically equivalent to the Schrödinger's wave mechanics formulation, provides advantages for some problems (Feynman, 2005).

According to Feynman, the propagator given by Eq. 2.11 can be written as (Neto, 2017)

$$K(q,t;q_0,t_0) \propto \sum_{\text{all paths}} \exp\left(\frac{iS}{\hbar}\right), \tag{2.13}$$

where S is the classical action. But how do we implement this sum of all paths? The answer has already been hinted at by Eqs. 2.11 and 2.12. Let

$$S(j,j-1) := \int_{t_{j-1}}^{t_j} L(x,\dot{x})dt, \tag{2.14}$$

where, for the integral to be evaluated, a path must be prescribed such that we can think of $S(j,j-1)$ fixing a path joining the points (q_{j-1},t_{j-1}) and (q_j,t_j). With that said, note that

$$\begin{aligned}\prod_{j=1}^{N} \exp\left[\frac{iS(j,j-1)}{\hbar}\right] &= \exp\left[\frac{i}{\hbar}\sum_{j=1}^{N} S(j,j-1)\right] \\ &= \exp\left[\frac{iS(N,1)}{\hbar}\right]\end{aligned}$$

represents the contribution of one path to the transition amplitude. Note that this is possible since, as mentioned above, S is the classical action[5]. From the correspondence given by Eq. 2.12, it can be assumed

[5] For quantum operators A and B, the identity $e^A e^B = e^{A+B}$ is true if, and only if, $[A,B] = 0$.

that

$$K(q_j,t_i,q_{j-1},t_{j-1}) = A\exp\left[\frac{iS(j,j-1)}{\hbar}\right], \tag{2.15}$$

where A is a constant to be fixed. Since, as it was already commented above, the measure ε of the intervals $[t_{j-1},t_j]$ are taken to be infinitesimal, we can approximate the path taken in Eq. 2.15 as a straight line, such that we obtain

$$K(q_j,t_i,q_{j-1},t_{j-1}) = A\exp\left[\frac{i\varepsilon L}{\hbar}\right], \tag{2.16}$$

where the Lagrangian must be taken as

$$L = L\left(\frac{q_j+q_{j-1}}{2},\frac{q_j-q_{j-1}}{\varepsilon},t_j\right). \tag{2.17}$$

We could naively plug Eq. 2.15 into Eq. 2.11 with the purpose of writing

$$\begin{aligned} K(q,t;q_0,t_0) &= \int\prod_{i=1}^{N-1}dq_i\prod_{j=1}^{N}A\exp\left[\frac{iS(j,j-1)}{\hbar}\right] \\ &= A^N\int\prod_{i=1}^{N-1}dq_i\exp\left[\frac{iS(N,1)}{\hbar}\right]. \end{aligned}$$

However, the action on the RHS of the last equation is not the classical one. Instead, we can be more precise and take the limit of $N\to\infty$ and $\varepsilon\to 0$ while keeping $N\varepsilon$ fixed. In this way, we obtain (Neto, 2017)

$$K(q,t;q_0,t_0) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0 \\ N\varepsilon\ \text{fixed}}} A^N\int\prod_{i=1}^{N-1}dq_i\exp\left[\frac{iS}{\hbar}\right], \tag{2.18}$$

where S, now, can be considered as the classical action in some calculations, although it is formally defined on a lattice.

The constant A can be obtained through the requirement that the path integral formulation gives the same transition amplitudes as the Schrödinger's formulation (Neto, 2017). It is given by

$$A = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}, \tag{2.19}$$

such that Eq. 2.18 becomes

$$K(q,t;q_0,t_0) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0 \\ N\varepsilon \text{ fixed}}} \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{N/2} \int \prod_{i=1}^{N-1} dq_i \exp\left[\frac{iS}{\hbar}\right]. \tag{2.20}$$

What we have done, essentially, is: in order to compute the path integral over the infinite dimensional Hilbert space in which the states $|x,t\rangle$ lie, we chose a finite dimensional subspace through the process of partitioning the interval $[t_0,t]$, such that we obtained a lattice. By taking the continuum limit of this lattice to obtain the whole Hilbert space, we claimed that the propagator could be defined in terms of the classical action. This procedure involves some deep mathematical questions that lie beyond the scope of this text. For example, one can ask: does the limit always converge? Is the limit independent of the choice of the subspace? (Chen et al., 2019).

2.3 Feynman path integrals in phase space

Up to this point, we have developed the path integral formulation over configuration space in the Schrödinger picture. The extension of this machinery to the Hamiltonian approach can be done with little effort and it provides a way to generalize this formulation to field theory. Eq. 2.8 tells us that, by summing over the energy eigenstates, we can write, in the Heisenberg picture,

$$K(q,t;q_0,t_0) = \langle q|\exp\left(\frac{-iH(t-t_0)}{\hbar}\right)|q_0\rangle. \tag{2.21}$$

Therefore, considering an interval $[t_j,t_{j-1}]$ with measure ε, we have

$$K(q_j,t_j;q_{j-1},t_{j-1}) = \langle q_j|\exp\left(\frac{-i\varepsilon H}{\hbar}\right)|q_{j-1}\rangle. \tag{2.22}$$

The same trick behind the statement of Eq. 2.5 can be used to establish a basis of momentum states, such that

$$\mathrm{id} = \int dp|p\rangle\langle p|, \tag{2.23}$$

and, consequently, to establish the following relation:

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{i}{\hbar} qp\right). \tag{2.24}$$

Therefore, we can insert an identity into the RHS of Eq. 2.22 and obtain

$$\begin{aligned}\langle q_j|\exp\left(\frac{-i\varepsilon H}{\hbar}\right)|q_{j-1}\rangle &= \int dp_j \langle q_j|\exp\left(\frac{-i\varepsilon H}{\hbar}\right)|p_j\rangle\langle p_j|q_{j-1}\rangle \\ &= \int dp_j \exp\left(\frac{-i\varepsilon H(q_j,p_j)}{\hbar}\right)\langle q_j|p_j\rangle\langle p_j|q_{j-1}\rangle \\ &= \frac{1}{2\pi\hbar}\int dp_j \exp\left[\frac{i}{\hbar}\left(p_i(q_i-q_{i-1})-\varepsilon H(q_j,p_j)\right)\right]. \end{aligned} \tag{2.25}$$

Now, if we plug this last equation into 2.11 considering Eq. 2.22, we get

$$K(q,t;q_0,t_0) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0 \\ N\varepsilon \text{ fixed}}} \left(\frac{1}{2\pi\hbar}\right)^N \int \prod_{i=1}^{N-1} dq_i \prod_{i=1}^{N} dp_i \exp\left[\varepsilon \frac{i}{\hbar}\sum_{i=1}^{N}\left(p_i\frac{q_i-q_{i-1}}{\varepsilon} - H(q_j,p_j)\right)\right], \tag{2.26}$$

which may be rewritten in the more compact form,

$$K(q,t;q_0,t_0) = \int DqDp \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t} d\tau[p\dot{q} - H(q,p,\tau)]\right\}, \tag{2.27}$$

where

$$\int DpDq = \lim_{\substack{N\to\infty \\ \varepsilon\to 0 \\ N\varepsilon \text{ fixed}}} \left(\frac{1}{2\pi\hbar}\right)^N \int \prod_{i=1}^{N-1} dq_i \prod_{i=1}^{N} dp_i \tag{2.28}$$

and $\dot{q} = dq/d\tau$.

Not only the Hamiltonian formulation of the path integral is easier to generalize to field theory, as mentioned in the beginning of this section, it is also a generalization of the path integral, as it was formulated in the previous section. For systems whose Hamiltonian is quadratic in the momentum, one may safely start from the Lagrangian formulation of the path integral, since it can be recovered from the Hamiltonian formulation through integration over momenta[6]. However, for other, more general systems where the Hamil-

[6] This point will be clarified further in the first section of chapter 3

tonian's dependence on the momentum is not so simple, the Hamiltonian formulation provides the proper description of such system (Neto, 2017).

2.4 Functional integrals in field theory

The path integral formulation, when applied to field theory, provides a way to quantize fields which is mathematically equivalent to the canonical quantization procedure usually introduced in quantum field theory textbooks. It is also requires less computation steps when compared to the canonical quantization procedure in most of the physical systems. However, the transition of the path integral formulation from non-relativistic quantum mechanics to field theory requires some preparation.

In quantum field theory, the observable of main interest is the S-matrix, since it encodes all the information about the scattering processes of interacting fields and is used to compare a quantum field theory's predictions with experiments in high-energy particle colliders (Schwartz, 2014). Except in some exactly solvable models, the calculation of the S-matrix is perturbative and relies on the N point Green's functions. As an example, the two point Green's function $G^{(2)}$ for a scalar field ϕ is defined as (Gomes, 2002)

$$G^{(2)}(x_1,x_2) = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle, \tag{2.29}$$

where $|0\rangle$ is the vacuum state of the theory and T is the called the *time ordering operator* and is the defined in the following way:

$$T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y) \text{ if } x^0 > y^0 \\ \phi(y)\phi(x) \text{ if } y^0 > x^0 \end{cases} . \tag{2.30}$$

The path integral formulation, in this context, provides an alternative approach to calculate N point Green's functions for a field theory, as we will see further.

Let $Q(t_j)$ be the position operator of which $|q_j,t_j\rangle$ is an eigenvector. In the sense described above for scalar fields, we want to consider the matrix elements $\langle q,t|Q(t_j)|q_0,t_0\rangle$, since the scalar field ϕ is the corresponding configuration space observable in the field theory. Regarding Eqs. 2.9 and 2.11, we have

$$\langle q,t|Q(t_j)|q_0,t_0\rangle = \int dq_1 dq_2 ... dq_{N-1} \langle q,t|q_{N-1},t_{N-1}\rangle ... \\ \times ... \langle q_{j+1},t_{j+1}|Q(t_j)|q_j,t_j\rangle ... \langle q_1,t_1|q_0,t_0\rangle. \tag{2.31}$$

Hence, considering Eq. 2.20 and the development made in the last section leading to Eq. 2.27, we can write

$$\langle q,t|Q(t_j)|q_0,t_0\rangle = \int DqDp\; q_j \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t} d\tau[p\dot{q} - H(p,q)]\right\}. \tag{2.32}$$

Now, consider the quantity $\langle q,t|Q(t_k)Q(t_j)|q_0,t_0\rangle$. If $t_k > t_j$, we can apply the same line of reasoning behind the last equation and write

$$\langle q,t|Q(t_k)Q(t_j)|q_0,t_0\rangle = \int DqDp\; q_k q_j \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t} d\tau[p\dot{q} - H(p,q)]\right\}. \tag{2.33}$$

If $t_k < t_j$, we would need to account for the commutation relation between $Q(t_j)$ and $Q(t_k)$. Nevertheless, as one can observe in this case, $\langle q,t|Q(t_j)Q(t_k)|q_0,t_0\rangle$ equals the RHS of Eq. 2.33, so it is convenient to write

$$\langle q,t|TQ(t_k)Q(t_j)|q_0,t_0\rangle = \int DqDp\; q_k q_j \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t} d\tau[p\dot{q} - H(p,q)]\right\}, \tag{2.34}$$

where T is the time ordering operator defined analogously, as in Eq. 2.30, for the operators $Q(t_j)$. The extension of this expression for N operators $Q(t_1), Q(t_2), ..., Q(t_N)$ follows trivially.

Eq. 2.34 already looks similar to Eq. 2.29, except that it still does not characterize a vacuum-vacuum transition amplitude. To proceed, we need to introduce a function $J(\tau)$ for $t_0 < \tau < t$ which acts like an external source. This function plays an essential role in the application of the path integral formalism to field theory, providing a way to obtain the aforementioned vacuum-vacuum transition amplitude through functional derivation. Define

$$\langle q,t|q_0,t_0\rangle^J \equiv \int DqDp \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t} d\tau[p\dot{q} - H + Jq]\right\}. \tag{2.35}$$

Note that

$$\lim_{J\to 0}\frac{\delta^N}{\delta J(t_{i_1})\delta J(t_{i_2})...\delta J(t_{i_N})}\langle q,t|q_0,t_0\rangle^J=\left(\frac{i}{\hbar}\right)^N\langle q,t|TQ(t_{i_1})Q(t_{i_2})...Q(t_{i_N})|q_0,t_0\rangle, \tag{2.36}$$

where it is important to recall that, in order to obtain this, one must use a lattice to express this last equation in the form of 2.26.

Now, there is an important result that further enables us to make the connection with field theory. It states that (Abers; Lee, 1973)

$$\lim_{\substack{t''\to\infty\\ t'\to-\infty}}\frac{\langle q'',t''|q',t'\rangle^J}{\exp[-iE_0(t''-t')]\phi_0^*(q')\phi_0(q'')}=\int dq dq_0\ \phi_0^*(q,t)\langle q,t|q_0,t_0\rangle^J\phi_0(q_0,t_0). \tag{2.37}$$

where E_0 is the ground state energy, $\phi_0(q,t)$ is an energy eigenfunction with eigenvalue E_0. Note that the RHS of Eq. 2.37 is the vacuum-vacuum amplitude and hence our quantity of interest to obtain the correspondence with the Green's functions of field theory. To make this idea more explicit, let's define

$$W[J]\equiv\int dq dq_0\ \phi_0^*(q,t)\langle q,t|q_0,t_0\rangle^J\phi_0(q_0,t_0). \tag{2.38}$$

Considering Eq. 2.36, we have that

$$\begin{aligned}\lim_{J\to 0}\frac{\delta^N W[J]}{\delta J(t_{i_1})\delta J(t_{i_2})...\delta J(t_{i_N})}&=\left(\frac{i}{\hbar}\right)^N\int dq dq_0\ \phi_0^*(q,t)\phi_0(q_0,t_0)\\ &\quad\times\langle q,t|TQ(t_{i_1})Q(t_{i_2})...Q(t_{i_N})|q_0,t_0\rangle\\ &=\left(\frac{i}{\hbar}\right)^N\langle 0|TQ(t_{i_1})Q(t_{i_2})...Q(t_{i_N})|0\rangle.\end{aligned} \tag{2.39}$$

It is convenient to make the definition

$$\mathscr{N}\equiv\frac{1}{\exp[-iE_0(t''-t')]\phi_0^*(q')\phi_0(q'')}, \tag{2.40}$$

such that from Eq. 2.37, we have

$$W[J]=\lim_{\substack{t''\to\infty\\ t'\to-\infty}}\mathscr{N}\langle q'',t''|q',t'\rangle^J. \tag{2.41}$$

Regarding Eq. 2.35, we can rewrite this last equation as

$$W[J] = \mathscr{N} \int DqDp \exp\left\{\frac{i}{\hbar}\int_{-\infty}^{+\infty} d\tau[p\dot{q} - H + Jq]\right\}. \tag{2.42}$$

By the limit imposed in Eq. 2.37, it is possible to realize that $\mathscr{N}$, as it appears in Eq. 2.42, is infinite. However, since it is independent of J and does not appear in Eq. 2.39, it is safe to ignore this fact (Neto, 2017). Therefore, Eq. 2.42 is ready to be generalized to fields. In this case, $W[J]$ is also called the *generating functional* and is defined, for a scalar field ϕ, as

$$W[J] = \mathscr{N} \int D\phi D\Pi \exp\left\{\frac{i}{\hbar}\int d^4x[\Pi\dot{\phi} - \mathscr{H} + J\phi]\right\}, \tag{2.43}$$

where Π is the conjugate momentum field and $\mathscr{H}$ is the Hamiltonian density[7]. As one can notice, Eq. 2.43 is basically the generalization of Eq. 2.42 with the phase space coordinates replaced by their corresponding fields. The N point Green's functions, then, is given by applying the LHS of Eq. 2.39 to the definition 2.43:

$$\lim_{J\to 0} \frac{\delta^N W[J]}{\delta J(t_{i_1})...\delta J(t_{i_N})} = i^N \langle 0|\phi(x_{i_1})...\phi(x_{i_N})|0\rangle, \tag{2.44}$$

where we set $\hbar = 1$, as we will do from now on.

As one can see, Eq. 2.44 does not depend on the troublesome constant $\mathscr{N}$, and hence it can be ignored, as it was already mentioned. Eq 2.44 contains both connected and disconnected Feynman diagrams, but, as it happens in the canonical quantization approach, the disconnected diagrams may be easily dismissed, since they don't contribute to the interaction processes (Neto, 2017).

2.5 Partition function and thermal states

One of the advantages of the functional integral formulation of quantum field theory is that it enables us, with little effort given what was already developed in this text, to work with fields exhibiting thermal properties. This is achieved through a correspondence between the partition function from statistical mechanics and the generating functional introduced in the last section. This formulation is an invaluable

[7] In field theory, the Hamiltonian is given as a functional of the field and its conjugate momentum: $H = \int d^3x \mathscr{H}(\phi, \Pi)$.

tool for particle physicists and indispensable when dealing with gauge theories (Kapusta; Landshoff, 1989). Nevertheless, our main focus here will be on the scalar field theory. The discussion in this section will be central to the discussion in the next chapter.

First, let us recast some of the formalism already presented in the last sections using fields. As it is usual in quantum field theory, one promotes the classical field ϕ and its conjugate momentum Π to operators $\hat{\phi}$ and $\hat{\Pi}$. Their eigenstates, in the Schrödinger's picture at time $t=0$, can be written as

$$\hat{\phi}(x,0)|\phi\rangle = \phi(x)|\phi\rangle \tag{2.45}$$

and

$$\hat{\Pi}(x,0)|\Pi\rangle = \Pi(x)|\Pi\rangle. \tag{2.46}$$

We may also consider $|\phi\rangle$ and $|\Pi\rangle$ as bases and write, in the same way as we did for Eqs. 2.5 and 2.23,

$$\mathrm{id} = \int d\phi(x)|\phi\rangle\langle\phi| \tag{2.47}$$

and

$$\mathrm{id} = \int d\Pi(x)|\Pi\rangle\langle\Pi|. \tag{2.48}$$

The analogous of Eq. 2.24, in this context, is given by

$$\langle\phi|\Pi\rangle = \exp\left(i\int d^3x\Pi(x)\phi(x)\right). \tag{2.49}$$

Therefore, it is possible to reproduce all the steps done in the last sections for fields in an analogous way. Let $|\phi_0\rangle$ and $|\phi_f\rangle$ denote two eigenstates of the operator $\hat{\phi}(x,0)$ with eigenvalues $\phi_0(x)$ and $\phi_f(x)$, respectively. The transition amplitude for state $|\phi_0\rangle$ at time $t=0$ to be found in state $|\phi_f\rangle$ at time t_f is given by

$$\begin{aligned}\langle\phi_f|e^{-iHt_f}|\phi_0\rangle = \mathscr{N}\int D\phi D\Pi \\ \times\exp\left\{i\int_0^{t_f} dt\int d^3x\Big[\Pi\dot{\phi} - \mathscr{H}\Big]\right\},\end{aligned} \tag{2.50}$$

where the integral over momenta, Π, is unbounded and the integral over ϕ runs over all paths that have, as its end points, ϕ_0 at $t=0$ and ϕ_f at $t=t_f$.

When dealing this statistical mechanics, one is more interested in systems which return to their original state after a specific time (Kapusta; Landshoff, 1989). Then, the amplitude we must consider has the boundary condition $\phi_0=\phi_f$, i.e. the periodic boundary condition. The partition function, in this context, is defined as (Das, 1997)

$$Z=\mathrm{Tr}e^{-\beta H}=\int d\phi_0\langle\phi_0|e^{-\beta H}|\phi_0\rangle. \tag{2.51}$$

Therefore, we may interpret the transition amplitude in Eq. 2.50 as the partition function if we make a Wick rotation in time,

$$t_f=-i\beta, \tag{2.52}$$

together with a change of variable $it=\tau$, and let the integration over ϕ to go over all paths which are periodic, i.e., fields that repeat themselves after imaginary time $\tau=\beta$. Then, we have

$$Z(\beta)=\mathscr{N}\int_{\phi(\tau=0)=\phi(\tau=\beta)}D\phi D\Pi\exp\left\{\int_0^\beta d\tau\int d^3x\left[i\Pi\dot{\phi}-\mathscr{H}\right]\right\}. \tag{2.53}$$

In cases where the Hamiltonian density $\mathscr{H}$ has the momentum dependence in the form of a quadratic term, it is possible to do the integration over momenta immediately (Bernard, 1974), since the momentum dependent part of the exponential will be of the form of a Gaussian integral, as we will see in the next chapter.

3 PATH INTEGRAL APPROACH TO THE UNRUH EFFECT

> You could cook your steak by accelerating it (if the minor problem, that a temperature of 300°C requires an acceleration of about $10^{24} cm/sec^2$, did not make the technique somewhat impractical). (Unruh, 1990, p. 108)

The usual approach to demonstrate the Unruh effect is via the canonical quantization method, as can be seen in Unruh (1976), Crispino, Higuchi and Matsas (2008), Mukhanov (2007) and Wald (1994). However, the path integral approach is equivalent and more sophisticated. Furthermore, it is more easily generalized to interacting field theories. The existence of the Unruh effect, through this method, is verified through the demonstration that the Green's function in the accelerated frame at the Unruh temperature is equivalent to the corresponding Green's function in the inertial frame at zero temperature. Hence, it's necessary that the partition function in the accelerated frame at Unruh temperature be equivalent to the generating functional in the inertial frame at zero temperature. In this chapter, we provide a review of this approach to the Unruh effect based in the work of Unruh and Weiss (1984).

In section 3.1, we derive the generating functional for a real, free, massive scalar field based on the books by Neto (2017) and Greiner and Reinhardt (2013). The accelerated frame, also called the *Rindler frame*, is then presented in section 3.2 following the steps of Rindler himself as laid out in his special relativity book (Rindler, 1991) and seminal paper (Rindler, 1966). Finally, in section 3.3, we present, in the Rindler frame, the generating functional obtained in section 3.1 and show their equivalence. In this chapter, units are chosen such that $c = \hbar = k = 1$.

3.1 Generating functional of the Klein-Gordon field

Let us consider a real, free, massive Klein-Gordon field whose Lagrangian density is given by

$$\mathscr{L} = \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2). \tag{3.1}$$

From the Lagrangian density, we can write the action explicitly in coordinates $x^\mu = (t, x, y, z)$:

$$S = \frac{1}{2}\int d^4x \Big[\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\Big], \tag{3.2}$$

where $(\nabla\phi)^2 = \nabla\phi\cdot\nabla\phi$. The conjugate momentum can be easily calculated to be

$$\Pi = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \tag{3.3}$$

$$= \dot{\phi}. \tag{3.4}$$

The Hamiltonian density is defined as the Legendre transformation $\mathscr{H} = \Pi\dot{\phi} - \mathscr{L}$, while the Hamiltonian of this system is defined to be (Neto, 2017)

$$H = \int d^3x \mathscr{H}. \tag{3.5}$$

It can be written explicitly in coordinates as

$$H = \frac{1}{2}\int d^3x \Big[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\Big]. \tag{3.6}$$

Now, we can obtain the generating functional for the Klein-Gordon field, in inertial coordinates, by plugging Eqs. 3.6 and 3.4 into Eq. 2.43 with the source $J = 0$, for simplicity. We have

$$W[0] = \mathscr{N}\int_{\phi \text{ periodic}} D\phi D\Pi \exp\left\{-\frac{i}{2}\int d^4x \Big[\Pi^2 + (\nabla\phi)^2 - 2\Pi\dot{\phi} + m^2\phi^2\Big]\right\}. \tag{3.7}$$

Note that, by completing the square, the integral over the momenta can be cast in the following form:

$$\int D\Pi \exp\left\{-\frac{i}{2}\int d^4x \Big[\Big(\Pi - \dot{\phi}\Big)^2 - \dot{\phi}^2\Big]\right\}. \tag{3.8}$$

Through the change of variables $\Pi' = \Pi - \dot{\phi}$, we can cast Eq. 3.8 into a product of Gaussian integrals,

$$\exp\left\{\frac{i}{2}\int d^4x\dot{\phi}^2\right\}\lim_{N\to\infty}\int\prod_{i=1}^{N} d\Pi_i' \exp\left\{-\frac{i}{2}\int d^4x(\Pi_i')^2\right\} \tag{3.9}$$

To evaluate this, one can divide spacetime into N cells "i" of volume ε^4. After that, one just recovers the continuum by taking the limit $\varepsilon^4 \to 0$, $N \to \infty$ (Neto, 2017):

$$\exp\left\{\frac{i}{2}\int d^4x\dot{\phi}^2\right\}\lim_{N\to\infty}\int\prod_{i=1}^{N}d\Pi_i'\exp\left\{-\frac{i}{2}\varepsilon^4\sum_{i=1}^{N}(\Pi_i')^2\right\}$$
$$= \exp\left\{\frac{i}{2}\int d^4x\dot{\phi}^2\right\}\lim_{N\to\infty}\left(-\frac{2\pi i}{\varepsilon^4}\right)^{N/2}. \tag{3.10}$$

Note that the constant $(-2\pi i/\varepsilon^4)^{N/2}$ goes to infinity as $N \to \infty$. However, we can just absorb this constant in $\mathscr{N}$ which, as already mentioned in section 2.4 of chapter 2, is already infinite and has no relevance to quantum field theory. Hence the generating functional, as given by Eq. 3.7, becomes

$$W[0] = \mathscr{N}\int D\phi\,\exp(iS), \tag{3.11}$$

where S is the action given by Eq. 3.2.

One can observe that the oscillating exponential in Eq. 3.11 presents a problem for its convergence. To circumvent this, one can rotate the integration contour in the complex time plane as long as the rotated contour does not cover any poles. In this case, this analytic continuation in t is independent of the angle of rotation and, for simplicity, a Wick rotation is sufficient, i.e., $t = -i\tau$. By doing this, Eq. 3.11 becomes

$$W[0] = \mathscr{N}\int D\phi\,\exp(-S_E), \tag{3.12}$$

where

$$S_E = \frac{1}{2}\int d^4x\Big[(\nabla_4\phi)^2 + m^2\phi^2\Big] \tag{3.13}$$

is called the *Euclidean action*, and $(\nabla_4\phi)^2 = (\partial_\tau\phi)^2 + (\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z\phi)^2$. Note that, now, the exponential falls off and convergence is guaranteed. Eq. 3.12 is called the *Euclidean generating functional* and is often denoted as $W_E[J]$ (Greiner; Reinhardt, 2013).

3.2 The Rindler frame

In chapter 2, we discussed how the generating functional from quantum field theory can be used to describe thermal states and, in the previous section, we discussed the generating functional for a real, massive, free Klein-Gordon field in inertial coordinates. Since the Unruh effect is about the frame dependence of the vacuum's particle content, it is also necessary to be able to describe the generating functional in the accelerated frame. To obtain the accelerated frame, let us first consider the law of transformation of velocities: let S and S' be inertial frames with relative velocity V in the x direction, also let u_x be the velocity of a particle as measured in the frame S; the relation between u_x and the particle's velocity as measured in S', u'_x, is given by (Hobson; Efstathiou; Lasenby, 2006)

$$u'_x = \frac{u_x - V}{1 - u_x V}. \tag{3.14}$$

Now, suppose that the particle moves with uniformly varying velocity u along the x direction of the inertial frame S and let S' be the instantaneous rest frame[1] of such particle. Hence, we can consider the frame S' as travelling with speed u, the particle's velocity, with respect to frame S. By setting $V = u_x$ in Eq. 3.14 and differentiating it with respect to t, we have

$$\frac{du_x}{dt} = \frac{du'_x}{dt}(1 - u_x^2). \tag{3.15}$$

From the Lorentz transformations, we can obtain (Rindler, 1991)

$$\frac{dt'}{dt} = \gamma(u_x)(1 - u_x V), \tag{3.16}$$

where $\gamma(u_x)$ is the gamma factor from special relativity taking into account that the speed of S' with respect to S is u_x. Therefore, through the previous equation, and recalling that we set $V = u_x$, we can write Eq. 3.15 as

$$\frac{du'_x}{dt'} = (1 - u_x^2)^{-3/2}\frac{du_x}{dt}, \tag{3.17}$$

[1] It is a frame of reference where the observer is at rest at each instant of its proper time.

where du'_x/dt' may be identified with the particle's proper acceleration as measured in its instataneous rest frame S'. Let $a \equiv du'_x/dt'$, then Eq. 3.17 may be written in the simpler form

$$a = \frac{d}{dt}[\gamma(u_x)u_x]. \tag{3.18}$$

If we integrate the previous equation, choosing the initial condition $u(0) = 0$, we have

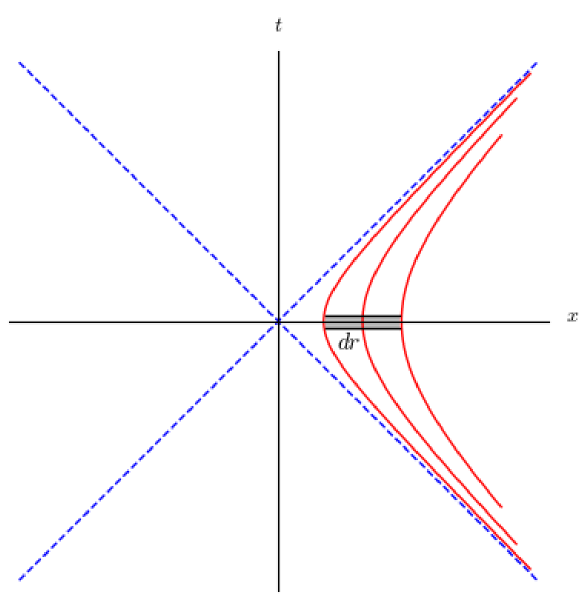
$$at = \frac{u_x}{\sqrt{1-u_x^2}}. \tag{3.19}$$

By solving for u_x and integrating again, we obtain the following equation of motion

$$x^2 - t^2 = \frac{1}{a^2}, \tag{3.20}$$

which describes hyperbolae on the xt plane parametrized by the proper acceleration a.

Figure 3.1 – An accelerated rigid rod (space filled among the hyperbolae) representing different accelerated observers (the red hyperbolae).



Source: Author (2022)

However, instead of Eq. 3.20, consider the alternative, more general,

$$x^2 - t^2 = r^2. \tag{3.21}$$

This equation describes accelerated observers with acceleration $1/r$. One may interpret the variable r as a proper location on a rigid rod in accelerated motion (see Fig. 3.1). Then, one can obtain the following coordinates from Eq. 3.21:

$$x = r\cosh\eta; \tag{3.22}$$

$$t = r\sinh\eta; \tag{3.23}$$

the coordinates (η, r) are the so called Rindler coordinates (Rindler, 1966) and describe the proper frame of an accelerated observer. Note that the Minkowski metric in Rindler coordinates becomes

$$ds^2 = r^2 d\eta^2 - dr^2 - dy^2 - dz^2. \tag{3.24}$$

Therefore when $r = 1/a$, i.e., when considering an observer with constant proper acceleration a, we have that the accelerated observer's proper time τ is η/a. So it is often convenient to make the further coordinate change $\eta = a\tau$ (Crispino; Higuchi; Matsas, 2008), such that Eqs. 3.22 and 3.23 become

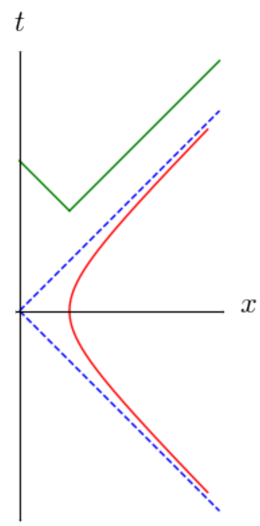
$$x = r\cosh(a\tau); \tag{3.25}$$

$$t = r\sinh(a\tau); \tag{3.26}$$

where $r > 0$ and $-\infty < \tau < \infty$. These coordinates cover just a quarter of Minkowski space — the *right Rindler wedge*, which is the region $x > |t|$. The *left Rindler wedge* can be covered by considering $r < 0$.

As one can note, the accelerated observer experiences an event horizon (see Fig. 3.2), i.e., signals emitted from the region $t > |x|$ cannot reach the observer. If one considers an observer with proper acceleration a, it is possible to realize that it cannot measure distances greater than a^{-1} in the direction opposite

Figure 3.2 – A signal (in green) above the dashed line $x = t$ (in blue) cannot be observed by the accelerated observer (in red).



Source: Author (2022)

to the observer's acceleration[2]. Furthermore, signals emitted from accelerated observers in the left Rindler wedge also cannot be detected in the right Rindler wedge, and vice versa.

3.3 The Unruh effect

The central theme of this approach to the Unruh effect relies on the claim that imaginary time Green's functions, which may be obtained from the imaginary time generating functional (see Sec. 2.5), are also thermal (Gibbons; Perry, 1993). For that, it is necessary to obtain the imaginary time generating functional in the Rindler frame. The action 3.2, in the Rindler coordinates 3.25 and 3.26, can be obtained to be (Unruh; Weiss, 1984)

$$S^R = \frac{1}{2}\int_{RR} dr d\tau d^2x_\perp (ar)\left[\frac{1}{(ar)^2}(\partial_\tau\phi)^2 - (\partial_r\phi)^2 - (\nabla_\perp\phi)^2 - m^2\phi^2\right], \tag{3.27}$$

where RR denotes the right Rindler wedge, $d^2x_\perp \equiv dydz$ and $(\nabla_\perp\phi)^2 = (\partial_y\phi)^2 + (\partial_z\phi)^2$. From this action, one may also obtain the Hamiltonian

$$H^R = \frac{1}{2}\int_{RR} dr d^2x_\perp (ar)\left[(\Pi^R)^2 + (\partial_r\phi)^2 + (\nabla_\perp\phi)^2 + m^2\phi^2\right], \tag{3.28}$$

[2] There is a coordinate singularity at $r = 0$.

where

$$\Pi^R = \frac{1}{ar}(\partial_\tau \phi) \tag{3.29}$$

is the conjugate momentum in Rindler coordinates.

By reproducing the exact same steps presented in section 3.1, one may obtain the generating functional in Rindler coordinates by plugging Eq. 3.28 into Eq. 2.53 and immediately integrating over the momenta. This procedure yields

$$Z^R(\beta) = \mathscr{N} \int_{\phi(\tau=0)=\phi(\tau=\beta)} D\phi \exp\Big[-S_E^R(\beta)\Big], \tag{3.30}$$

where

$$S_E^R(\beta) = \frac{1}{2}\int_0^\beta d\tau \int dr d^2x_\perp (ar)\left[\frac{1}{ar}(\partial_\tau \phi)^2 + (\partial_r \phi)^2 + (\nabla_\perp \phi)^2 + m^2\phi^2\right] \tag{3.31}$$

is the Euclidean action in Rindler coordinates. Now, if we perform the change of integration variables

$$t_e = r\sin(a\tau) \tag{3.32}$$

and

$$x_e = r\cos(a\tau), \tag{3.33}$$

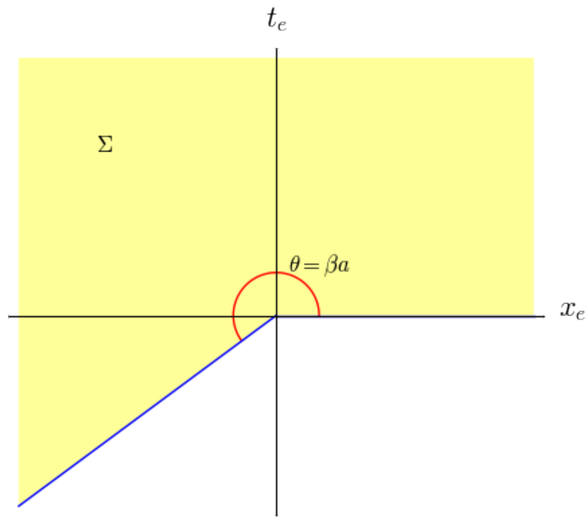
requiring that $r > 0$ and $0 \le a\tau \le \beta a \le 2\pi$ for the transformation to be single valued, we obtain

$$\begin{aligned} Z^R(\beta) &= \mathscr{N} \int D\phi \exp\left\{-\frac{1}{2}\int_\Sigma d^4x \Big[(\nabla_4 \phi)^2 + m^2\phi^2\Big]\right\} \\ &= \mathscr{N} \int D\phi \exp(-S_E), \end{aligned} \tag{3.34}$$

where $d^4x = dt_e dx_e d^2x_\perp$. If $\beta = 2\pi/a$, the region Σ covered by $t_e, x_e, x_\perp$ is the full $\mathbb{R}^4$ (see Fig. 3.3). Note that the RHS of 3.34 is just the Euclidean generating functional in inertial coordinates. From this, one can immediately show that the Green's functions in the Rindler frame at inverse temperature $\beta = 2\pi/a$ is equivalent to the Green's functions in the inertial frame at zero temperature, i.e.

$$\frac{\mathrm{Tr}[e^{-\beta H^R} T_\tau(\phi(r_1,\tau_1)...\phi(r_n,\tau_n))]}{\mathrm{Tr}(e^{-\beta H^R})} = \langle 0|T_t(\phi(x_1,t_1)...\phi(x_n,t_n))|0\rangle, \tag{3.35}$$

Figure 3.3 – Region covered by the change of integration variables given by Eqs. 3.32 and 3.33.



Source: Author (2022)

where T_τ and T_t are the time ordering operators with respect to the Rindler frame and the inertial frame, respectively (Unruh; Weiss, 1984).

Equation 3.35 shows that the physics as seen by a Rindler observer at temperature $T = a/2\pi$ is equivalent with that seen by a Minkowski observer in his vacuum state (Unruh; Weiss, 1984). It is important to note that, to demonstrante this, the change of integration variables, given by Eqs. 3.33 and 3.32, requires β to be the inverse Unruh temperature in order to map $Z^R(\beta)$ to the generating function in inertial coordinates at zero temperature properly. For the application of this approach to the Unruh effect to other scenarios, such as in the context of a Kleinian geometry, one can expect that the corresponding change of integration variables must satisfy the same, or analogous, requirements. This will be discussed further in chapters 5 and 6.

4 TOPICS IN KLEINIAN RELATIVITY

> I see that it is by no means useless to travel, if a man wants to see something new (Verne, 1873, p. 46)

The standard geometric treatment of general relativity lies upon the Lorentzian signature, $(+---)$ or $(-+++)$, since it provides a natural physical interpretation of the time and space dimensions. One can even argue that the Lorentzian signature is special, being the only signature for the metric of spacetime allowing the existence of the universe as we know it (Tegmark, 1997).

Nevertheless, different metric signatures have been explored. As an example, the Euclidean signature, $(++++)$ or $(----)$, is used in quantum field theory as a way of avoiding convergence problems in path integrals (Greiner; Reinhardt, 2013) and of giving a treatment of the thermal properties of fields (Kapusta; Landshoff, 1989) (see also Secs. 2.5 and 3.1). The Euclidean signature has also been used as a way to avoid the singularity in the beginning of the universe, as stated by the "No boundary condition" proposed by Hartle and Hawking (1983).

The remaining possibility, considering a 4 dimensional space, is the signature $(++--)$, which has been called the *Kleinian signature* in some of the physics community (Alty, 1994; Barrett et al., 1994; Figueiredo et al., 2016; Alves-Júnior; Barreto; Moraes, 2021), but also goes by the name of *neutral signature* by some authors of the mathematical physics community (Law, 1991; Hall; Kırık, 2018).

Another interesting context where the Kleinian signature appears is in the realm of metamaterials (Smolyaninov; Narimanov, 2010). The behavior of light in such media gives rise to effective geometries and may be studied from the perspective of pseudo-Riemannian geometry (Sátiro; Moraes, 2006). These systems can be used as analog models of gravity, and the emergence of the Kleinian signature in some cases enables phenomena such as optical black holes (Fumeron et al., 2015). Also, differently from the Minkowski space of Special Relativity, the kinematics of a relativistic particle in a Kleinian geometry has the momentum of a particle bounded, while its speed is not (Alves-Júnior; Barreto; Moraes, 2021).

In this chapter, we discuss some of the relativistic implications of a spacetime with a metric with a Kleinian signature $(++--)$ — a *Kleinian space*. In section 4.1, we derive a general expression for the generators of isometries of the Kleinian space and give an explicit example of a particular generator, further called the *Klein boost*. We then proceed to discuss some implications of the Klein boost in section 4.2. The dynamics of particles in Kleinian space is then discussed in section 4.3. Finally, we derive a Kleinian space analog of the accelerated frame in section 4.4. For this chapter, we chose units such that $c = 1$.

4.1 Symmetries of Kleinian space

To understand relativity in Kleinian space, it is instructive, first, to consider the isometries of the flat metric with a Kleinian signature. In this case, they come from $SO(2,2)$ except for the translational symmetries. Isometries may be obtained from the Killing equation (Wald, 1984)

$$\nabla_b \xi_a + \nabla_a \xi_b = 0, \tag{4.1}$$

where ξ^a is a Killing vector field. Since we are currently interested in flat space, the covariant derivatives become ordinary partial derivatives and the Killing equation may be rewritten as

$$\partial_b \xi_a + \partial_a \xi_b = 0. \tag{4.2}$$

The solutions for this system of differential equations give us the generators of the Lie algebra $\mathfrak{so}(2,2)$ plus translations — the Kleinian analog of the Poincaré algebra.

To solve Eq. 4.2, it is useful to take the second derivative of it,

$$\partial_c \partial_b \xi_a + \partial_c \partial_a \xi_b = 0, \tag{4.3}$$

and cycle the indices twice, such that we get three equations

$$\partial_c \partial_b \xi_a + \partial_c \partial_a \xi_b = 0 \tag{4.4}$$

$$\partial_a \partial_c \xi_b + \partial_a \partial_b \xi_c = 0 \tag{4.5}$$

$$\partial_b \partial_a \xi_c + \partial_b \partial_c \xi_a = 0 \tag{4.6}$$

By adding Eqs. 4.4 and 4.5 and subtracting Eq. 4.6, we get

$$\partial_c \partial_a \xi_b = 0, \tag{4.7}$$

which means that, if we choose coordinates x^α, the components of the Killing vector field must have the form

$$\xi_\alpha = A_\alpha + M_{\alpha\beta}x^\beta, \tag{4.8}$$

where A_α and $M_{\alpha\beta}$ are constants. Hence, the Killing equation 4.2, in coordinates, becomes

$$M_{\alpha\beta} = -M_{\beta\alpha}. \tag{4.9}$$

Note that Eqs. 4.8 and 4.9 tell us that there are 4 independent constants A_α and 6 $M_{\alpha\beta}$, giving 10 independent constants, in total, characterizing the Killing vector fields given by Eq. 4.8. The constants $M_{\alpha\beta}$ can be represented as anti symmetric matrices and comprise a representation of $\mathfrak{so}(2,2)$.

Now, consider the flat metric with Kleinian signature in coordinates $x^\alpha = (t,x,y,z)$

$$\begin{aligned} ds^2 &= g_{\alpha\beta}dx^\alpha dx^\beta \\ &= dt^2 + dx^2 - dy^2 - dz^2. \end{aligned} \tag{4.10}$$

To get explicit examples of the generators, it suffices to set one of the constants equal to 1 while keeping the rest equal to 0. An example which will be useful in the next section is obtained by setting $M_{01} = 1$. Then, the generator is given by

$$\begin{aligned} (\xi^{(01)}) &= g^{\alpha\nu}M_{\nu\beta}x^\beta\partial_\alpha \\ &= g^{00}M_{01}x^1\partial_0 - g^{11}M_{10}x^0\partial_1 \\ &= x\partial_t - t\partial_x, \end{aligned} \tag{4.11}$$

which is just the generator of rotations of the xt plane, a symmetry which is not present in Minkowski space. The corresponding element of the Lie group $SO(2,2)$ may be obtained through the exponential map

$$K^{(01)} = \exp\left[\psi\xi^{(01)}\right], \tag{4.12}$$

where ψ is a parameter and we chose to denote the group element by K to enforce that we are currently dealing with symmetries in Kleinian space. The action of the group element in the coordinate $x^0 = t$ is given by

$$
\begin{aligned}
K^{(01)}t &= \exp\left[\psi \xi^{(01)}\right] t \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\psi(x\partial_t - t\partial_x)\right]^n t \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \psi^{2n+1} x + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \psi^{2n} t \\
&= t\cos(\psi) + x\sin(\psi).
\end{aligned}
\tag{4.13}
$$

The same calculation for the coordinate x yields

$$
K^{(01)}x = x\cos(\psi) - t\sin(\psi).
\tag{4.14}
$$

Finally, $K^{(01)}y = y$ and $K^{(01)}z = z$.

The corresponding $K^{(01)}$ in usual Minkowski space is the Lorentz boost in the x direction. Therefore, it is possible to observe the sharp contrast between the two cases: Minkowskian and Kleinian geometries. In Minkowski space, the orbits of the Lorentz boost Killing vector field are hyperbolae and the coordinate transformations are given in terms of hyperbolic functions (Wald, 1994). In Kleinian space, the orbits of the "Kleinian boost" $K^{(01)}$ are circles and the coordinate transformations are given in terms of ordinary trigonometric functions. As we will discuss further, these differences will manifest themselves in the analogue accelerated frame in the Kleinian space.

4.2 Kleinian transformations and some consequences

Although Eqs. 4.13 and 4.14 characterize a rotation of the xt plane, we may also interpret K as a "boost" in Kleinian space if we view the additional timelike coordinate x, in this context, as a kind of anisotropy rather than another time dimension. In this sense, the parameter ψ may be interpreted as the "rapidity", in direct analogy with special relativity. Its relation with the velocity β is given by

$$
\tan(\psi) = \beta.
\tag{4.15}
$$

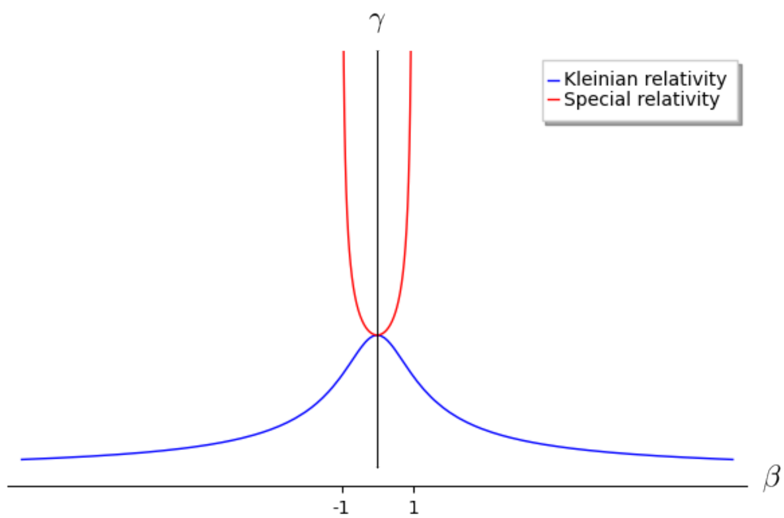
Hence, Eqs. 4.13 and 4.14 become

$$t' = \frac{t \pm x\beta}{\sqrt{1+\beta^2}}, \tag{4.16}$$

$$x' = \frac{x \mp t\beta}{\sqrt{1+\beta^2}}, \tag{4.17}$$

where $t' = K^{(01)}t$ and $x' = K^{(01)}x$. Note that, in this notation, we have $y' = y$ and $z' = z$. These transforma-

Figure 4.1 – Comparison between the gamma factors in Kleinian and Special relativity.



Source: Author (2022)

tions are called *Kleinian transformations* (Alves-Júnior; Barreto; Moraes, 2021) and are the Lorentz transformations analogs in Kleinian space. A comparison between the analog "gamma factor" $\gamma = 1/\sqrt{1+\beta^2}$ in Eqs. 4.16 and 4.17 and its counterpart in special relativity (Fig. 4.1) shows that for $|\beta| \ll 1$, both behave approximately in the same way.

Two puzzling consequences of the Kleinian transformations are the length dilation and time contraction, the opposite of what special relativity implies. This difference is mainly the consequence of the difference between the gamma factors. Let S and S' denote two reference frames with relative speed β; also, let ℓ_0 denote the proper length of a rigid rod at rest measured by an observer in S'. One can show, using Eq.

4.17, that an observer in the reference frame S will measure a length ℓ given by

$$\ell = \ell_0 \sqrt{1+\beta^2}. \tag{4.18}$$

Now, if we let $\Delta\tau$ be the interval of time between two events happening at the same location in S', as measured by an observer at rest there, the interval of time Δt measured in S can be obtained from Eq. 4.16 and is given by

$$\Delta t = \frac{\Delta\tau}{\sqrt{1+\beta^2}}. \tag{4.19}$$

Velocities also transform differently in Kleinian relativity. Through Eqs. 4.16 and 4.17, one can obtain

$$\frac{dx'}{dt'} = \frac{\frac{dx}{dt} - \beta}{1 + \frac{dx}{dt}\beta}. \tag{4.20}$$

Note that the transformation of velocities law in Kleinian relativity implies the possibility of superluminal speed, which goes against the second postulate of special relativity. Nevertheless, for $\beta \ll 1$, the Newtonian limit is recovered.

4.3 Particle dynamics

New particle dynamics arises from the Kleinian geometry. Because of the difference in the signature of the metric when compared to the Minkowskian geometry, the energy and momentum of a particle also behave differently. To see this, let us consider the Lagrangian of a free, massive particle,

$$L = -m\sqrt{g_{ab}u^a u^b}, \tag{4.21}$$

where u^a is the particle's 4-velocity. The 4-momentum can be calculated directly from the last equation via the expression $p_a = \partial L / \partial u^a$ (Landau; Lifshitz, 1975). Hence,

$$p_i = -\frac{m g_{ij} u^j}{\sqrt{g_{ab}u^a u^b}};\ i,j = 1,2,3. \tag{4.22}$$

Expanding the last equation in the coordinates x, y and z, we have

$$p_x = -\frac{mv_x}{\sqrt{1+v^2}}, \tag{4.23}$$

$$p_y = \frac{mv_y}{\sqrt{1+v^2}}, \tag{4.24}$$

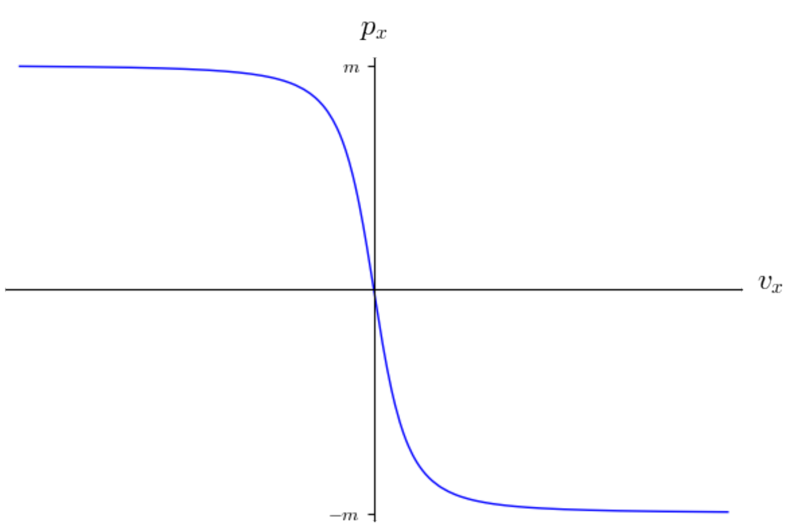
$$p_z = \frac{mv_z}{\sqrt{1+v^2}}, \tag{4.25}$$

where v_x, v_y and v_z are the coordinate components of the particle's 3-velocity and $v^2 = g_{ij}u^j = v_x^2 - v_y^2 - v_z^2$; $i, j = 1, 2, 3$. Note that if the particle is moving in the x direction ($v_y = v_z = 0$), then

$$p_x = -\frac{mv_x}{\sqrt{1+v_x^2}}. \tag{4.26}$$

Therefore, in the limit $v_x \to \pm\infty$, we have that the momentum p_x tends to $\mp m$,i.e., the momentum in the x

Figure 4.2 – Momentum in the x direction in Kleinian space



Source: Author (2022)

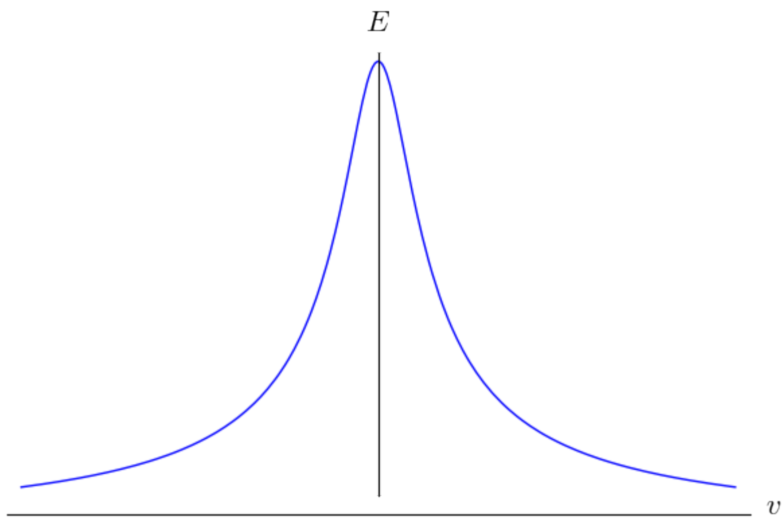
direction is bounded (see Fig 4.2). This is yet another strange consequence of Kleinian relativity since, in special relativity, speed is bounded by the speed of light while momentum is unbounded. Moreover, it can be seen that, in the x direction, the momentum points in the velocity's opposite direction.

Both in classical mechanics and special relativity, the energy of a system is defined through the Hamiltonian, which may be obtained through the Legendre transformation $H = p\dot{q} - L$ (Landau; Lifshitz, 1975). Hence

$$E = \frac{m}{\sqrt{1+v^2}}. \tag{4.27}$$

Note that the energy, as a function of speed v, has the same shape of the gamma factor of Kleinian space

Figure 4.3 – Energy E as a function of speed v.



Source: Author (2022)

(Fig. 4.3). This means that, as the particle's speed increases, energy tends to zero. By setting the momentum to $p^\mu = (E, p_x, 0, 0)$, we may obtain the energy conservation equation through $p_\mu p^\mu = m^2$, whose explicit form is

$$E^2 + p^2 = m^2. \tag{4.28}$$

These results have already been discussed in (Wang, 2016) where an "exotic matter" with negative mass and reversed momentum was investigated. It was also observed that the behavior of this kind of matter share some similarities with dark energy.

4.4 Accelerated frame

Following the analogies with special relativity, we can also obtain the analog accelerated frame in Kleinian space. In special relativity, the Rindler metric (Rindler, 1966),

$$ds^2 = r^2 d\tau^2 - dr^2 - dy^2 - dz^2, \tag{4.29}$$

can be derived by considering a family of parametrized observers whose trajectories can be described by the following coordinates:

$$x = r\cosh(a\tau); \tag{4.30}$$

$$t = r\sinh(a\tau); \tag{4.31}$$

with $r > 0$. The coordinates (τ, r, y, z) comprise the Rindler coordinates and describe observers with proper acceleration $1/r$, i.e., dependant on proper position. In order to keep the same form of the metric above, Eq. 4.29, while only changing the signature, one must consider the following family of parametrized observers:

$$x = r\cos(a\tau); \tag{4.32}$$

$$t = r\sin(a\tau); \tag{4.33}$$

again, with $r > 0$ and proper acceleration $1/r$. With these coordinates, we obtain the Kleinian metric,

$$ds^2 = r^2 d\tau^2 + dr^2 - dy^2 - dz^2, \tag{4.34}$$

associated with the accelerated frame.

However, for the unconvinced reader, it is also possible to obtain the accelerated frame, in this context, from the relativistic dynamics in Kleinian space. Consider a frame S which is at constant acceleration a in the x direction (e.g. the frame of an accelerating rocket). An observer at rest with respect to S has 4-velocity

$$u^\mu = (1, 0, 0, 0). \tag{4.35}$$

But, since the frame is accelerating, the observer at rest should still measure an acceleration in the x direction with an ideal accelerometer. It's 4-acceleration is given by

$$w^{\mu} = (0, a, 0, 0) \tag{4.36}$$

From Eqs. 4.35 and 4.36, we can obtain the following relations:

$$u^{\mu} u_{\mu} = 1; \tag{4.37}$$

$$w^{\mu} w_{\mu} = a^2; \tag{4.38}$$

Note that we also obtain the orthogonality between 4-velocity and 4-acceleration,

$$w^{\mu} u_{\mu} = 0. \tag{4.39}$$

Relations 4.37 and 4.38 yield

$$(u^0)^2 + (u^1)^2 = 1 \tag{4.40}$$

and

$$\left(\frac{du^0}{d\tau}\right)^2 + \left(\frac{du^1}{d\tau}\right)^2 = a^2, \tag{4.41}$$

from where we can obtain the following equations:

$$\frac{du^1}{d\tau} = a\sqrt{1-(u^1)^2}; \tag{4.42}$$

$$(u^0)^2 = 1-(u^1)^2. \tag{4.43}$$

The differential equation 4.42 has the solution

$$u^1(\tau) = \sin(a\tau + C). \tag{4.44}$$

By plugging this solution into Eq. 4.43, we obtain

$$u^0(\tau) = \pm\cos(a\tau + C). \tag{4.45}$$

We can rewrite Eqs. 4.44 and 4.45 explicitly in coordinates as

$$\frac{dx}{d\tau} = \sin(a\tau + C) \tag{4.46}$$

and

$$\frac{dt}{d\tau} = \pm\cos(a\tau + C), \tag{4.47}$$

respectively. Hence, integrating the above equations with respect to τ yields

$$x(\tau) = x_0 - \frac{1}{a}\cos(a\tau + C), \tag{4.48}$$

$$t(\tau) = t_0 \pm \frac{1}{a}\sin(a\tau + C). \tag{4.49}$$

Note that we have the freedom to choose a minus sign on Eq. 4.49. Then, to choose $C = \pi$ is equivalent to setting the initial conditions $u^1(0) = 0$ and $u^0(0) = 1$. Now we may set the initial conditions $x(0) = 1/a$ and $t(0) = 0$ such that Eqs. 4.48 and 4.49 become

$$x(\tau) = \frac{1}{a}\cos(a\tau), \tag{4.50}$$

$$t(\tau) = \frac{1}{a}\sin(a\tau). \tag{4.51}$$

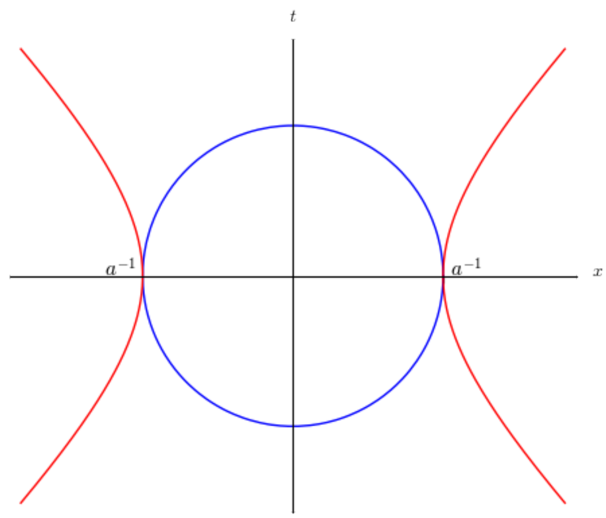
Recall that, as we have mentioned earlier, the proper acceleration in Rindler coordinates depends on the inverse of proper position, i.e., the proper acceleration is a if proper position is $1/a$. Since we have derived the above coordinates by considering an observer with proper acceleration a, we may generalize it to arbitrary proper acceleration by writing

$$x(\tau) = r\cos(a\tau) \tag{4.52}$$

and

$$t(\tau) = r\sin(a\tau), \tag{4.53}$$

Figure 4.4 – Comparison between the worldlines, with proper acceleration a, of Rindler observers (in red) and analog accelerated observers in Kleinian space (in blue).



Source: Author (2021)

thus reproducing Eqs. 4.32 and 4.33. Note that instead of hyperbolas, which are the worldlines of Rindler observers, we have circles in Kleinian space (see Fig. 4.4). Furthermore, the worldlines of the analog accelerated observers in Kleinian space are closed timelike curves, which, albeit allowed in general relativity, pose paradoxes such as the possibility of the accelerated observer going back in time and killing itself (Thome, 1992).

It is also worth noting that the proper time τ of the analog accelerated observer in Kleinian space is compactified, being restricted to the interval $(0, 2\pi/a]$. This compactification, which resembles the one appearing in finite temperature field theory (see Sec. 2.5 of Chap. 2), raises some questions about the characterization of a thermal field theory in Kleinian space, as will be discussed in the next chapter.

5 UNRUH EFFECT IN AN EFFECTIVE KLEINIAN GEOMETRY

> "You don't need to accept everything as true, you only have to accept it as necessary.""Depressing view," said K. "The lie made into the rule of the world." (Kafka, 1925, p. 125)

In this chapter, we use the path integral approach to the Unruh effect, presented in chapter 3, in an attempt to derive the Unruh effect in Kleinian space. For that, we use the analog accelerated frame presented in chapter 4 as an analog Rindler frame in Kleinian space. In section 5.1, we show how an effective geometry arises from an alternative interpretation of light rays in some kinds of anisotropic media. In section 5.2, we propose an effective metric of Kleinian signature, based on what is discussed in section 5.1, and use it to develop the generating functional of a massless, free scalar field in such effective Kleinian geometry. Finally, in section 5.3, we provide an attempt at a derivation of the Unruh effect in Kleinian space and discuss the problems that appear.

5.1 Effective geometry from anisotropic media

Fermat's principle provides a natural way to interpret paths of light rays in anisotropic media as geodesics in an effective geometry. Indeed, the trajectory of light rays may be obtained through the minimization of the integral

$$\mathscr{F} = \int_A^B N d\ell, \tag{5.1}$$

where A and B are the trajectory's endpoints and N is the refractive index of the media, which may vary or not. In both Riemannian geometry and General Relativity, the geodesic between two points on a manifold can be obtained by minimizing the integral (Nakahara, 2018)

$$I = \int_A^B ds, \tag{5.2}$$

where $ds = \sqrt{g_{ab}dx^a dx^b}$. Therefore, the following identification can be made (Sátiro; Moraes, 2006)

$$N d\ell = \sqrt{g_{ab}dx^a dx^b}; \tag{5.3}$$

the optical properties of the medium are thus interpreted as an effective geometry characterized by the metric g_{ab} (Sátiro; Moraes, 2006). As an example, in a hyperbolic liquid crystal equipped with a disclination, light experiences an effective metric given by (Fumeron et al., 2015)

$$ds^2 = c^2dt^2 - \varepsilon_0 dr^2 - \varepsilon_e r^2 d\phi^2 - \varepsilon_0 dz'^2, \tag{5.4}$$

where the permittivities ε_0 and ε_e are allowed to take different signs, e.g. $\varepsilon_0 > 0$ and $\varepsilon_e < 0$. Supposing this is the case, one can safely reescale and transform these coordinates[1] such that one obtains, in canonical form,

$$ds^2 = c^2dt^2 + dx^2 - dy^2 - dz^2, \tag{5.5}$$

where one can see that the anisotropy of the material manifests itself as a Kleinian signature in the effective metric.

5.2 The scalar field in an effective Kleinian geometry

Consider an effective metric given by

$$ds^2 = dt^2 - \varepsilon_1 dx^2 - \varepsilon_2 dy^2 - \varepsilon_2 dz^2, \tag{5.6}$$

where $\varepsilon_1 < 0$ and $\varepsilon_2 > 0$. Since the effective metric comes from the interpretation of light rays in anisotropic media as geodesics in a manifold, we will consider, for the description of light in this context, the massless scalar field Lagrangian,

$$\mathscr{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \tag{5.7}$$

whose field dynamics models electromagnetic waves in such media (Smolyaninov; Narimanov, 2010). The equation of motion corresponding to this Lagrangian is

$$g^{\mu\nu} \partial_\mu \partial_\nu \phi = 0, \tag{5.8}$$

whence, assuming plane wave solutions, we can obtain the following dispersion relation:

$$\omega^2 = \frac{k_x^2}{\varepsilon_1} + \frac{k_y^2 + k_z^2}{\varepsilon_2}, \tag{5.9}$$

[1] One can make $x = \sqrt{\varepsilon_0} r \cosh\left(\sqrt{-\varepsilon_e/\varepsilon_0}\phi\right)$, $y = \sqrt{\varepsilon_0} r \sinh\left(\sqrt{-\varepsilon_e/\varepsilon_0}\phi\right)$ and $\sqrt{\varepsilon_0} z' = z$ (Fumeron et al., 2015).

where k_x,k_y and k_z are components of the wave vector. One may note that Eq. 5.9 is precisely the dispersion relation of extraordinary waves in hyperbolic metamaterials (Poddubny et al., 2013). Furthermore, by reescaling the coordinates — $\sqrt{\varepsilon_1}x \to x$, $\sqrt{\varepsilon_2}y \to y$ and $\sqrt{\varepsilon_2}z \to z$ — we get a flat metric with Kleinian signature,

$$ds^2 = dt^2 + dx^2 - dy^2 - dz^2. \tag{5.10}$$

From Eqs. 5.7 and 5.10, the action is given explicitly by

$$S_K = \frac{1}{2}\int d^4x \Big[\dot{\phi}^2 + (\partial_x\phi)^2 - (\nabla_\perp\phi)^2\Big], \tag{5.11}$$

where the subscript K denotes "Kleinian". By plugging this action into Eq. 2.43, with $J = 0$, one may note the generating functional in Kleinian space reads[2]

$$W_K[0] = \mathscr{N}\int D\phi \exp\left\{\frac{i}{2}\int d^4x \Big[\dot{\phi}^2 + (\partial_x\phi)^2 - (\nabla_\perp\phi)^2\Big]\right\}. \tag{5.12}$$

One may note that Eq. 5.12 can be obtained from Eq. 3.11 by a Wick rotation in the x coordinate and setting $m = 0$. Nevertheless, the connection between the Rindler frame and the analog accelerated frame in Kleinian geometry is not so clear. The analog Euclidean generating functional, in this Kleinian setting, is, in fact, of Lorentzian nature. If we make $t \to -it$, we get

$$W_L[0] = \mathscr{N}\int D\phi \exp\left\{-\frac{1}{2}\int d^4x \Big[\dot{\phi}^2 - (\partial_x\phi)^2 + (\nabla_\perp\phi)^2\Big]\right\} \tag{5.13}$$

$$= \mathscr{N}\int D\phi e^{-S_L}, \tag{5.14}$$

where the subscript L denotes "Lorentzian".

[2] The integration over the momenta may be done in the same way as it is shown in section 3.1 of chapter 3.

5.3 Problems with the Kleinian Unruh effect

By applying the coordinate transformations 4.32 and 4.33 to the action given by 5.11 we obtain

$$S_K^A = \frac{1}{2}\int_0^{2\pi/a} d\tau \int_0^{\infty} dr \int d^2x_\perp (ar)\left[\frac{1}{(ar)^2}(\partial_\tau\phi)^2 + (\partial_r\phi)^2 - (\nabla_\perp\phi)^2\right], \tag{5.15}$$

where the superscript A indicates the analog accelerated frame. In this case, the conjugate momentum is given by the same expression, Eq. 3.29, found in Sec. 3.3 of chapter 3, but it will be denoted here as Π_K^A. The Hamiltonian is given by

$$H_K^A = \frac{1}{2}\int r dr d^2x_\perp a\left[(\Pi_K^A)^2 - (\partial_r\phi)^2 + (\nabla_\perp\phi)^2\right]. \tag{5.16}$$

Hence,the generating functional with source $J = 0$ yields

$$W_K^A[0] = \mathscr{N}\int D\phi \exp\left[iS_K^A\right]. \tag{5.17}$$

Note that because time is already compactified due to the coordinate transformations 4.32 and 4.33, the generating functional resembles slightly the partition function for field theory at the Unruh temperature (Sec. 3.3 of chapter 3). Nevertheless, one must still uses the imaginary formalism to make the correspondence between the partition function and the generating functional. In doing so, one obtains

$$Z_K(\beta) = \mathscr{N}\int_{\phi(\tau=0)=\phi(\tau=\beta)} D\phi \exp\left[-S_L^A\right], \tag{5.18}$$

where we left β undetermined due to the analytic continuation of the time coordinate.

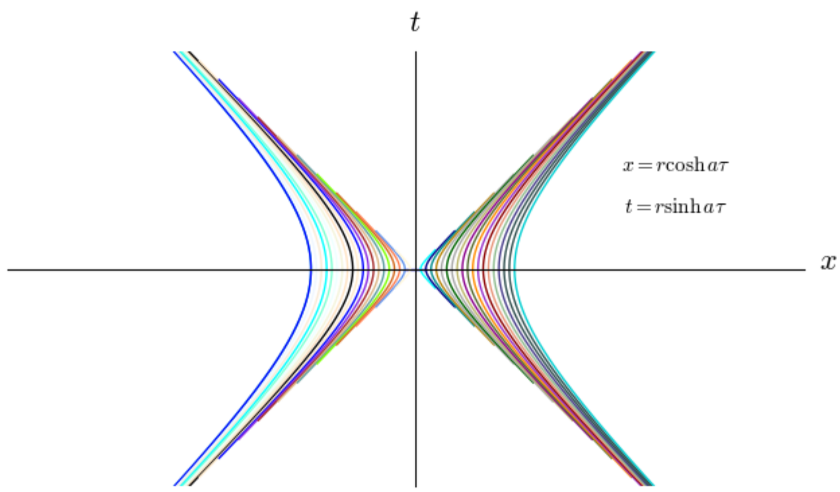
To obtain an analog Unruh effect in this context, it would be necessary to show that, through a change of integration variables, Eq. 5.18 is equivalent to Eq. 5.14. However, if one uses the change integration variables

$$t = r\cosh a\tau \tag{5.19}$$

and

$$x = r\sinh a\tau, \tag{5.20}$$

Figure 5.1 – Incomplete covering of the *xt* plane due to variables 5.19 and 5.20.



Source: Author (2022)

with $0 < r < \infty$ and $0 < \tau < \beta$, one obtains

$$Z_K(\beta) = \mathscr{N} \int D\phi e^{-S_L|_{RR}}, \tag{5.21}$$

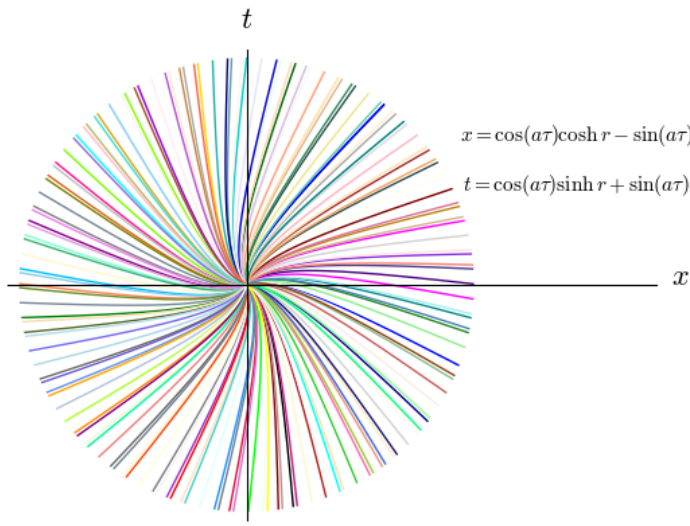
i.e., the partition function of the field in the analog accelerated frame at inverse temperature β is equivalent to the Lorentzian generating functional of the field in the inertial frame restricted to a portion of the right Rindler wedge (see Fig. 5.1). This goes in the opposite way considering what happens in the usual path integral approach to the Unruh effect, as shown in chapter 3. This also raises the question: is the dynamics of a particle in Kleinian space restricted by this portion of the right Rindler wedge? If no, then the procedure used here is wrong or inadequate; if yes, then there may exist an analog Unruh effect in Kleinian space.

We emphasize, again, the fact that the analog accelerated frame possesses a compactified time dimension (as already mentioned in chapter 4). This, together with periodic boundary conditions, are characteristics of the Matsubara formalism for describing finite temperature field theories and is summarized in the KMS condition (Das, 1997)

$$\langle A(t)A(t')\rangle_\beta = \langle A(t')A(t+i\beta)\rangle_\beta, \tag{5.22}$$

where $A(t)$ is a Heisenberg operator. But, as was already shown, the imaginary time formalism gives rise to a Lorentzian action, instead of an Euclidean action. Moreover, the necessary transformation to show the

Figure 5.2 – Covering of the *xt* plane due to variables 5.23 and 5.24 with $a\beta = 2\pi$.



Source: Author (2022)

equivalency between the partition function in the analog accelerated frame and the generating functional in the analog inertial frame restricts the action in the latter. In addition, considering the change of integration variables given by 5.19 and 5.20, there is no obvious condition enabling the determination of β.

Let us consider the following change of integration variables:

$$x = \cos a\tau \cosh r - \sin a\tau \sinh r - \cos a\tau; \tag{5.23}$$

$$t = \cos a\tau \sinh r + \sin a\tau \cosh r - \sin a\tau. \tag{5.24}$$

These variables can be obtained through the application of the two dimensional rotation matrix, with rotation angle $a\tau$, to hyperbolas. Note that these variables cover the entire *xt* plane if $a\beta = 2\pi$ (see Fig. 5.2). However, if we apply this change of integration variables to Eq. 5.15, we don't recover Eq. 5.14. Hence, in one hand, we have variables 5.19 and 5.20 which reproduce Eq. 5.14 but don't cover the *xt* plane; on the other hand, we have variables 5.23 and 5.24 covering the *xt* plane but not reproducing 5.14. To be able to show the existence of the Unruh effect using this approach, one must be able to find variables which do both.

As a final remark, it is important to note that in order to show an analog of Eq. 3.35, i.e. the equivalence between the Green's function at Unruh temperature in the accelerated frame and the Green's function at zero temperature in the inertial frame, it iss necessary to show that the time ordering operators

T_τ and T_t are equivalent (Unruh; Weiss, 1984). Since the Kleinian space possesses two timelike dimensions, there does not appear to be a preferred way of defining the time ordering operators. For this matter, an investigation of the causal structure of the Kleinian space may be useful.

6 CONCLUSION AND FINAL REMARKS

In this work, we showed some relativistic implications of the Kleinian signature and applied the path integral approach to the Unruh effect in a space endowed with a Kleinian signature. As we have seen, the Kleinian space has some exotic consequences, such as time contraction, length dilation and bounded momentum in the direction of motion, which defies well known phenomena of special relativity. Furthermore, we have seen that the analog accelerated observer's worldline is a closed timelike curve and its frame exhibits a compactified time dimension. Also, contrary to what happens in Rindler space, the analog accelerated frame, in the context of a Kleinian signature, covers the entire Kleinian space.

However, the application of the path integral approach to the Unruh effect in Kleinian space proved to be a task far from trivial. We showed that, by following the steps of Unruh and Weiss (1984), we do not obtain the expected equivalence between the partition function in the analog accelerated frame at the Unruh temperature and the generating functional in the analog inertial frame at zero temperature. This result may hint at issues lying in fundamental aspects related with the quantization of fields in the Kleinian space. Since global hyperbolicity is not a property shared by the Kleinian space, it might be difficult to quantize fields in such a background if we consider the formal prescription given by Wald (1994).

To obtain the results mentioned above, we presented a brief introduction to the path integral formalism in quantum mechanics and quantum field theory; from the definition of the propagator in quantum mechanics to the application of path integrals in finite temperature field theory to describe fields in thermal baths. We then showed how the path integral formalism can be used to demonstrate the Unruh effect.

One of our main motivations for our enterprise here, as mentioned in the introduction, is the work by Smolyaninov (2019). On the one hand, if photons in such metamaterials behaves as if they were in a Kleinian geometry, and the Unruh effect is expected to occur in such setting, then it is reasonable to investigate, as we did, whether the Unruh effect exists in a Kleinian space at all, since this phenomenon is a direct consequence of quantum field theory.

On the other hand, the geometry "experienced" by a photon in such metamaterial is consequence of a mathematical correspondence and, maybe, it should not be taken to the letter. Indeed, after all, a photon in a metamaterial is in ordinary spacetime by consequence. Since a Kleinian signature exhibits two timelike dimensions, it may be too naive to accept that a photon in such geometry would behave as if it were entirely subject, for example, to something similar to the "two time physics" proposed by Bars (2000).

As mentioned by Alves-Júnior, Barreto and Moraes (2021), although some aspects of the Kleinian space may be mimicked by some metamaterials, one cannot expect all of its relativistic implications to be reproduced in laboratories. Hence, when one uses this kind of analog system to probe fundamental results of physics, such as the Unruh effect, one can raise the question: what are the limits of analog models of gravity? Which phenomena can, or can't, be reproduced? And if some analog relativistic phenomenon can't be reproduced in laboratories to the letter, does it still correspond to some other phenomenon in the material?

Concerning further steps in our research, it may be interesting, as already mentioned in chapter 5, to investigate if there are alternatives to Eqs. 5.19, 5.20,5.23 and 5.24 which: i) cover the entire Kleinian space; ii) determine the inverse temperature β; iii) reproduce the Lorentzian generating functional. In order for these alternatives to do the latter, the new variables $x(\tau,r)$ and $t(\tau,r)$ must satisfy the equations

$$\frac{1}{(ar)^2}\left(\frac{\partial x}{\partial \tau}\right)^2-\left(\frac{\partial x}{\partial r}\right)^2=1, \tag{6.1}$$

$$\frac{1}{(ar)^2}\left(\frac{\partial t}{\partial \tau}\right)^2-\left(\frac{\partial t}{\partial r}\right)^2=1 \tag{6.2}$$

and

$$\frac{1}{(ar)^2}\frac{\partial x}{\partial \tau}\frac{\partial t}{\partial \tau}=\frac{\partial x}{\partial r}\frac{\partial t}{\partial r}. \tag{6.3}$$

If these new variables exist, then an analog Unruh effect in Kleinian space may exist.

On a more fundamental ground, it may be useful to start from first principles and attempt a construction of a quantum field theory in a Kleinian background. One of the main obstacles for this is the Cauchy problem for ultra-hyperbolic differential equations, as already noted by Tegmark (1997). However, for the free, massless scalar field, it may be possible to do such construction under some constraints (Foster; Müller, 2010).

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